USING AN APOS FRAMEWORK TO UNDERSTAND TEACHERS' RESPONSES TO QUESTIONS ON THE NORMAL DISTRIBUTION

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ABSTRACT

This study is an exploration of teachers' engagement with concepts embedded in the normal distribution. The participants were a group of 290 in-service teachers enrolled in a teacher development program. The research instrument was an assessment task that can be described as an "unknown percentage" problem, which required the application of properties of the standard normal distribution curve. Responses to the task were analyzed using the Action, Process, Object, Schema (APOS) framework that specified a standardization and a probability layer of understanding. The success rates were 27% and 14% in the two questions, with most teachers experiencing problems in the probability layer because of a failure to link the probability values with the area covered by the curve.

***Keywords**: Statistics education research; In-service mathematics program; South Africa; APOS theory*

1. INTRODUCTION

Apartheid policies in South Africa led to many disparities in education which the new democratic government is trying to reduce. One of the anomalies of apartheid education was the variation in the teacher preparation among the different races. In the previous system, most white teachers had a university degree with a minimum of one year of tertiary mathematics, while black teachers were more likely to have a three-year college teaching diploma (Adler, 1997). However, a teacher with only a three-year diploma is now considered to be underqualified.

In an attempt to provide upgrading opportunities for the mainly black underqualified teacher, some universities offered an Advanced Certificate in Education (ACE) on a part-time basis over two years. However many teachers could not cope with the demands of some ACE programs with one study finding that only 44% of the underqualified teachers were able to pass within minimum time (Bansilal, 2012a). This study is located in an ACE program designed for practicing mathematics teachers who were either underqualified (79% of the cohort) or who were fully qualified but wanted to refresh their knowledge about teaching mathematics (21% of the cohort). The program included a unit on statistics. Although a concept like the normal distribution curve is not part of the school curriculum, it was considered as an important component of what Ball, Thames and Phelps (2008) refer to as *horizon knowledge*. This is an 'awareness of how mathematical topics are related over the span of mathematics included in the curriculum' (p. 403) and is one of the six domains that comprise their model of *mathematical knowledge for teaching*. Having knowledge of the horizon can help teachers make decisions about how to teach concepts like variation, distributions and other statistical topics. The unit also covered aspects of statistics such as central tendencies, grouped data, distributions, bivariate data, regression, probability concepts and probability distributions. The teachers in the program were located all around the KwaZulu-Natal provinces, with many of them in remote rural areas. Hence, the program was delivered at eight different sites, comprising 14 classes in total. The course was coordinated by two university lecturers who prepared the course notes,

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activities and assessments, while the actual teaching was carried out by 14 tutors who were supervised by the two lecturers. Because many of the teaching sites were not well-equipped, it was not possible for the teachers to have access to computers while studying for this unit, and so the teaching was based mainly on written notes and class activities and did not include any computer simulation activities or the use of any computer applets.

Data for this study were generated by the responses of the 290 participants to a task consisting of two "unknown percentage" questions (Watkins, Scheaffer, & Cobb, 2004). In unknown percentage problems, a given value is first transformed into an associated *z*-score by means of the processes of re-centering and rescaling. The next step is to identify the probability value associated with the *z*-score, and this value is then interpreted in terms of the standard normal distribution graph. The purpose of the study is to unpack the teachers' understanding of the concepts that are required in the problems and is based on the Actions, Process, Object, Schema (APOS) theory proposed by Dubinsky (1991).

It is hoped that the study will add to pedagogic knowledge of statistics educators by helping identify particular points that students may be likely to struggle in, when working with similar problem solving tasks in statistics. The use of the APOS framework is also a particular contribution to the field, and perhaps other statistics education researchers may be prompted to examine whether the APOS framework could be used to investigate students' concepts in other areas of statistics education.

2. LITERATURE REVIEW

Scholarship about the teaching and learning of statistics has grown dramatically (Zieffler, Garfield, Alt, Dupuis, Holleque, & Chang, 2008). Taking statistics education research as studies that could inform our understanding of the learning and teaching of statistics (Zieffler et al., 2008), the study reported in this paper is about teachers' understanding of concepts associated with the normal distribution curve.

In South Africa, the statistics component now accounts for 14%–18% of the mathematics curriculum in the Grades 10–12 band (DoBE, 2011). Similar to situations across the world, it is mathematics teachers who are responsible for its implementation. Gattuso and Ottavani (2011) comment that "teachers generally have no preparation for teaching statistics, little knowledge about statistics and almost never any training in statistics education" (p. 123). In a similar vein, Wessels and Nieuwoudt (2013) point out that some teachers may have taken statistics courses in their undergraduate degrees, but it is likely that such courses were procedurally rather than conceptually inclined, and hence many mathematics teachers lack the proficiency to apply the statistical knowledge to practical settings. The teachers in this study did not study any statistics during their teacher training and their proficiency in statistics was also limited.

In training secondary mathematics teachers the mathematical knowledge is often seen as more important and in some instances, "particularly if mathematics is seen in a formalistic view, this may even hinder their grasp of statistics" (Gattuso & Ottavani, 2011, p. 124). Burrill and Biehler (2011) note that when students study probability with a formal approach, they will learn formalisms without understanding the phenomena described by this mathematics. Wilensky (1997, p. 172) concurs that the traditional teaching of the concept of normal distribution which relies on formalism and macro-level summary statistics can lead to epistemological anxiety (the feeling of confusion and indecision students have when faced with different paths for solving a problem).

Reading and Canada (2011) are of the opinion that the study of distribution of data is complex, yet fundamental to statistical reasoning. A distribution can be defined as "the arrangement of values of a variable along the scale of measurement resulting in a representation of the observed frequencies or the theoretical probability of a range of variables of the variable" (Reading & Canada, p. 224). Researchers (Cohen & Chechile, 1997; Reading & Canada, 2011) point out that a key step in statistical reasoning is understanding the differences between a data distribution and a probability distribution. Empirical (data) distributions are what are seen in the data by way of frequencies of the variables. Unlike data, probability distributions are formal theoretical models used to describe the likelihood of a variable taking on a value or a range of values. It is this theoretical nature that brings out contrasts between probability and data and can help students develop stochastic ideas (Cohen & Chechile, 1997). Probability distributions stand "at the interface between the traditional study of probability and the traditional study of statistics" and therefore provide an opportunity to make strong connections between the two fields (Wilensky, 1997, p. 175). These authors recommend that despite the emphasis placed on hands-on data analysis and alternative methods for inference, the concept of probability distributions should form part of any introductory statistics courses.

One of the most well-known probability distributions is the normal distribution, which can model many natural and psychological phenomena (Batanero, Tauber, & Sanchez, 2004). It also offers a good approximation for other distributions and many statistical methods require the condition of random samples from normal distributions. However, some authors (Batanero et al., 2004; Pfannkuch & Reading, 2006) have expressed concern that there is little research investigating students' understanding of the normal distribution. In fact, such concern led to Pfannkuch and Reading (2006) setting up the November 2006 *Statistics Education Research Journal* as a special issue focused on reasoning about distributions. The authors outline possible research questions that could address various aspects of reasoning about distributions, including one about the "difficulties that students encounter when working with analyzing and interpreting distributions" (p. 5).

An important consideration in setting up a study to investigate students' conceptions of topics in statistics is that of the theoretical framework. One that has been used often in statistics education research is the SOLO (Structure of the Observed Learning Outcome) (Biggs & Collis, 1982). Reading and Reid (2006) used the SOLO taxonomy to elaborate on a hierarchy of reasoning about distribution, that consisted of prestructural, unistructural, multistructural and relational levels. Their hierarchy is "arranged with increasing sophistication in dealing with the key elements of distribution" and entails two cycles of levels, where the first is prerequisite to the second (p. 58). The first cycle involves understanding of key elements and then the second, more cognitively sophisticated level involves using those elements. Reading and Reid (2006) argue that without a well-developed understanding of variation, students' ability to reason about distribution will be hampered.

3. METHODOLOGY

The study utilizes an interpretive approach because the main goal was to understand the participants' interpretations of reality (Cohen, Manion, & Morrison, 2011) as revealed in their responses to a task based on the normal distribution curve. The participants were 290 practicing teachers who had enrolled in an in-service program designed to upgrade and retrain mathematics teachers in the secondary school. This article focuses on one of the four mathematics modules devoted to a study of introductory probability and statistics suitable for Grades 10-12 mathematics teachers. The task, consisting of two questions, was administered as part of a summative classroom assessment, which included questions from other sections of the module. This study is part of a larger one that looks at secondary mathematics teachers' knowledge of statistics (see also Bansilal, 2012b).

3.1. THE TASK

The tasks used an application of the properties of the standard normal distribution as their basis (see Table 1). When the distribution of a variable in a set of data is approximately normal, one can use the properties of the standard normal distribution curve to make inferences about the variable under discussion. In "unknown percentage" problems (Watkins et al., 2004), students first transform a given value into an associated z-score by re-centering and re-scaling. Thereafter students associate a probability with the z-score and interpret its value in terms of the graph, by working simultaneously with properties of the standard normal distribution and the properties of particular z-table values.

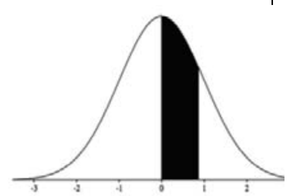
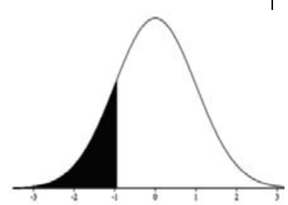
Table 1. Details of task

<table>
<tr><td colspan="3">A university entrance examination scores are scaled so that they are approximately normal. The mean is about 505 and the standard deviation is about 111.
1. Find the probability that a randomly selected student has a score below 400.
2. Find the probability that a randomly selected student has a score between 450 and 600.</td></tr>
<tr><td>Solution</td><td>Comments</td><td>Representation</td></tr>
<tr><td>1.
$z = \frac{x - \mu}{\sigma} = \frac{400 - 505}{111} = -0.95$

The p-value that corresponds to z = -0.95:
p = 0.3289.

Then
$P(x < 400)$
$= P(z < -0.95)$
$= 0.5 - 0.3289$
$= 0.1711$
$= 17.1\%$</td><td>Students are required to calculate the z-score corresponding to the x-value. They then read off the p-value corresponding to the z-score from the given z-table.

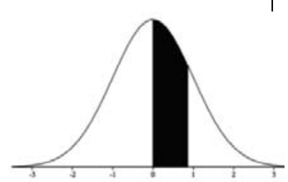
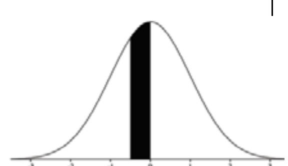
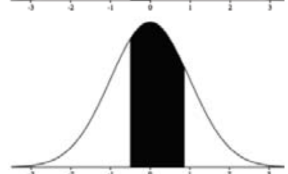
It then involves interpreting the p-value taken from the table to find P(x < 400). That is, students have to recognize that the shaded portion alongside is associated with the value p = 0.3289, or P(0< Z< 0.95), the convention followed by the z-table that was provided. Therefore, the area of the shaded portion is 0.3289.

The symmetry of the standard normal distribution implies that
P(0<Z<0.95) = P(–0.95<Z< 0).
In order to find the area of the shaded portion in the graph, students must subtract 0.3289 from 0.5.</td><td>
</td></tr>
<tr><td>2.The z-score that corresponds to 600:
$z = \frac{x-\mu}{\sigma} = \frac{600-505}{111} = 0.86$
The p-value that corresponds to z =0.86 is
$P_1 = 0.3051$

Finding the z-score that corresponds to 450:
$z = \frac{x-\mu}{\sigma} = \frac{450-505}{111} = -0.50$
Reading off P_2 = 0.1915 from the z-table

Finally: P(450 < X < 600) = $P_1 + P_2$= 0.3051 + 0.1915 = 0.4966 = 49.66%</td><td>Students read from the z-table and link the shaded portion to the probability P_1 = P(0 < Z <0.86) = 0.3051

Recognizing that the shaded portion is associated with the probability of P_2 =
P(–0.50 < Z < 0) = 0.1915

Recognizing that the area of the shaded portion is the probability
P(–0.50 < Z < 0.86) =
P(450 < X < 600)</td><td>

</td></tr>
</table>

In order to solve this problem, the teachers had access to a formula sheet that contained the standardization formula $z = \frac{x-\mu}{\sigma}$. The teachers could use scientific calculators. Different statistics textbooks use different tabulation values of the standard normal curve area for a given positive value z_0, like $P(0 < Z < z_0)$ or $P(Z < z_0)$ or $P(Z > z_0)$, where these are associated with the area of the corresponding sectors. In the lectures and the assessments, the z-table that the teachers used gave $P(0 < Z < z_0)$ for positive z_0.

Note that defining the random variable X is important for computing the probabilities associated with the random variable. In this case, the random variable is the entrance examination scores, which have a normal distribution.

3.2. FRAMEWORK FOR ANALYSIS

The framework that underpins this study is the well-known APOS (action, process, object, schema) theory of Dubinsky (1991), which is regarded by some researchers (Tall, 2011) as similar to the SOLO taxonomy because of its hierarchical description of students' progression in understanding concepts. APOS theory has been applied widely to many areas of mathematics, such as functions, calculus, discrete mathematics, linear algebra and fractions, amongst others (Dubinsky & McDonald, 2002). APOS theory has also been applied in statistics education, for example, Clark and Mathews (2003) looked at students' conceptions of mean and standard deviation. They found that traditional instruction in statistics seems to inhibit students from moving from a process to an object conception of standard deviation, and that it is very difficult for students to move beyond a strong process image of standard deviation.

In this study dealing with teachers' conceptions as revealed in a problem solving task, APOS theory was chosen because of its suitability to "analyze the knowledge that students' display when solving a specific activity at a particular moment in time" (Posani, Trigueros, Preciado, & Lozano, 2009, p. 2126).

APOS theory (Dubinsky, Weller, McDonald, & Brown, 2005) asserts that an individual deals with a mathematical situation by using certain mental mechanisms to build cognitive structures that are applied to the situation. The main mechanisms are called *interiorization* and *encapsulation* and the related structures are actions, processes, objects and schemas (ibid). According to the APOS framework, actions and processes are operations on previously established objects and each action needs to be interiorized into a process and then encapsulated into an object before being acted upon by other actions/processes. The structures are explained below.

Action In some tasks, students have to consider mathematical objects and perform actions on them, and reflect on their actions. A transformation is first conceived as an action when it is a reaction to stimuli which an individual perceives as external. It requires specific instructions and the need to perform each step of the transformation explicitly.

Process Other tasks have as a goal to incorporate those actions into algorithms or procedures. Reflection on how and why these work helps students abstract their main characteristics, take control over them and be able to use them flexibly. In APOS theory this is referred to as a *process conception*. This is when an action is interiorized into a mental process.

Object If one becomes aware of a process as a totality, realizes that transformations can act on that totality, and can actually construct such transformations, then we say that the individual has encapsulated the process into a cognitive object. The distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made.

Schema A schema is a more or less coherent collection of cognitive objects and internal processes for manipulating these objects. A schema could aid students to "understand, deal with, organize, or make sense out of a perceived problem situation" (Dubinsky 1991, p. 102).

As objects are operated on in further processes, the objects form layers of understanding. Sfard (1992, p. 70) asserts that the phrase 'process before object' refers to one individual cycle in the development of mathematical ideas, which "begins when a new idea is thematized and ends when it becomes a basis for a higher–level concept". This cycle adds another layer in the system of mathematical concepts, which may be seen as similar to the two cycles of understanding proposed by Reading and Reid (2006) in their study of reasoning about distribution.

In this scenario, the first layer is the standardization procedure and this forms a basis for the development of the higher-level concept of finding the probability of obtaining a z-score in a given range. In the analytic framework presented, the different levels of working within each of the two layers (standardization layer and the probability layer) are distinguished where understanding the first is a prerequisite for accessing the second layer. The action, process, and object structures in APOS theory are utilized to distinguish between the levels of understanding (or engagement with the problem) in each layer.

Standardization layer In order to find the percentage in an "unknown percentage problem", one first has to calculate the required z-score using the standardization formula

$$(1) \qquad z = \frac{x - mean}{SD} = \frac{x - \mu}{\sigma}$$

This stage is referred to as working in the standardization layer.

Action level engagement in the standardization layer. Actions are needed in order to change from an x-value in a normal distribution to a z-score of the standard normal distribution. At the action level, the person is limited to the use of the procedure (1) as a reaction to the external prompt of the x-value, and the transformation is done in a step by step manner, following the formula and may therefore be prone to computation errors. The student has not interiorized any of the properties of the elements in the standardization formula.

Process level engagement in the standardization layer. The action becomes interiorized into a process when it can be done in the mind, without the student having to work out each step separately. At this stage. the person can understand the elements of the process. A process understanding will enable a learner to recognize the relationship between z, x, μ and σ as well as to use the formula in a different manner, such as finding x when z is given. This is possible because at this stage a person can understand the reversal of a process, and thus manipulate equation (1).

Object level engagement in the standardization layer. An object understanding of this process will enable a student to see the signifier z as the result of, but separate from. the process which produced it. Only when the signifier z is recognized as an object which is a result of the calculation (1) can the person carry out further transformations such as relating it to the probability values. An object understanding will also enable learners to distinguish between and compare two objects z_1 and z_2 arising from a similar process, for example recognizing the positions of z_1 and z_2 relative to the mean and to each other as well as understanding how changes to values of x, μ and σ impact on z and vice-versa.

For Questions 1 and 2, the calculation of the z-scores required only a fluent application of the standardization procedure and hence this step did not require engagement higher than a process level. However an object level understanding of the z-score is necessary at the step which requires the manipulation of the band(s) of area associated with the p-value(s) in order to find the required probability as represented by the proportion of the combination of the bands of area. The interpretation of the band of area is dependent on the location of z and manipulation of the bands of area is made easier if one has access to an object perspective of the z-score. Those responses that indicated a struggle to combine the probability values associated with the given z-score, suggest an inability to view the z-score as a totality, hence limiting their interpretation of the p-values as bands of area which could be combined with other bands of area.

Probability layer Central to the standard normal distribution curve is the relationship between the z-score, the associated p-value and the band of area represented by the p-value. Each z-score on the standard normal distribution curve is associated with a probability or p-value, equivalent to the proportion of area covered by the curve in a specific band or interval, depending on the z-table that is used.

Action level engagement in the probability layer. At an action level, the student can read off the p-value from the z-table. The identification can be accomplished without necessarily having an understanding of the implications of the values. The figures in the z-tables act as external prompts.

Process level engagement in the probability layer. At a process level, the student has interiorized the relationship between the z-score and the p-value by being able to visualize the meaning of $z_0 = 2.5$, say, being associated with a p-value of 0.4938, using the probability table that gives $P(0 < Z < z_0)$. The student can interpret P = 0.4938 as representing the area of the curve between $z = 0$ and $z_0 = 2.5$. The student will also recognize for $-z_0 = -2.5$, the p-value= 0.4938 is the value of the area between $z = 0$ and $-z_0 = -2.5$. In other words. a person with a process understanding would be able to make use of the property $P(0 < Z < z_0) = P(-z_0 < Z < 0)$ because of the symmetry at 0.

Object level engagement in the probability layer. Engagement at an object level implies being able to view the z-score, the p-value associated with the z-score (as given by the particular standard normal probability table) and the associated band of the area as a single entity upon which further transformations can be made. This would imply that the student would be able to interpret the two probabilities (associated with two different z-scores) as representing two bands of area which can be manipulated or combined, allowing them to compare and consider the relationships between them. At this stage, a person would also be able to work equally comfortably with different standard normal probability tables because they would see the probability value as a different representation of the z-score and these have been encapsulated as one whole.

4. RESULTS

The teachers' responses to the two questions are described in the paragraphs that follow and these are discussed according to the levels in which they were categorized.

4.1. RESULTS FOR QUESTION 1

Pre-action-level responses in the standardization layer The blank responses (35) were coded as pre-action level because there was no evidence of any engagement with the standardization procedure. Responses containing an inappropriate formula were also taken as pre-action because there was no evidence of recognition of the correct formula. There were 32 responses where an inappropriate or incorrect algorithm retrieved from the formula sheet was applied. One such example was the use of the coefficient of variation formula as shown in Figure 1.

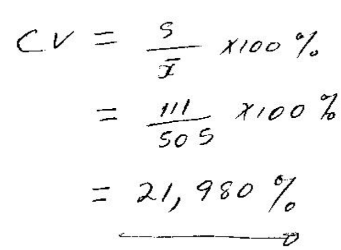


Figure 1. Example of a response (to Question 1) using an irrelevant formula.

Figure 1 shows a response where the coefficient of variation formula was used to try to find the probability of a student scoring below 400.

Action-level responses in the standardization layer There were 24 responses which applied the standardization formula incorrectly in Question 1 and this hindered further progress. These were considered as action level because there was evidence of some engagement with the standardization procedure even though it was not correct. An example of such a response is provided in Figure 2.

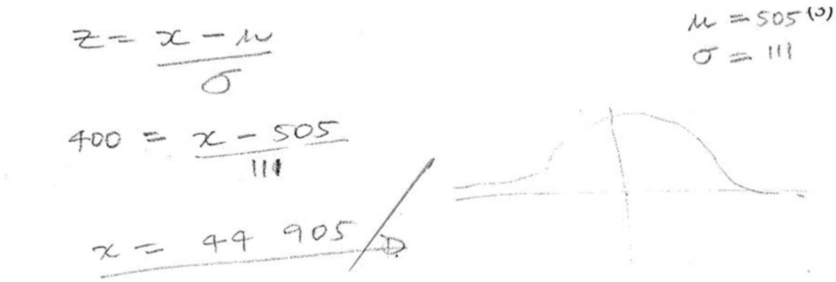


Figure 2. A response where the standardization formula is used incorrectly.

Figure 2 shows an action level response where the appropriate standardization formula is used incorrectly. In this response 400 is taken as the value of the *z*-score instead of considering it as the *x*-value.

Process-level responses in the standardization layer Many responses coped comfortably with the standardization layer, and calculated the value of *z*. For Question 1 there were 65 responses that were categorized as indicative of a process-level engagement in the first layer. However, some of these teachers did not go any further than calculating the *z*-score, while some took the *z*-score as the probability, for example, as shown in Figure 3.

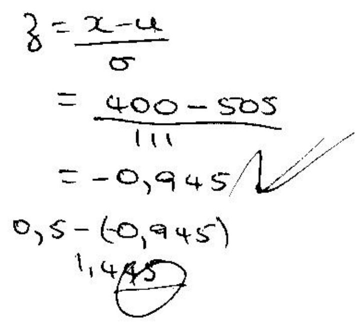


Figure 3. A response where the z-value was considered as the p-value.

Figure 3 shows a response of a teacher who first calculated the *z* value to be –0.945. Instead of then reading off a *p*-value from the table, she has subtracted –0.945 from 0.5, and arriving at an answer of 1.445. This is then left as her response for the required probability. This response indicated a process-level understanding of standardization.

Action-level responses in the probability layer There were 30 responses where the *z*-score was correctly calculated but the appropriate *p*-value corresponding to the *z*-value was not correctly identified. This was taken as indicative of an action-level of engagement at the probability layer (and a process-level engagement at the previous standardization layer). An example of such a response appears in Figure 4.

$$z = \frac{x - u}{\sigma}$$
$$= \frac{400 - 505}{111} = \frac{-105}{111} = -0{,}946$$
$$P(< -0{,}946) = -0{,}3389$$
$$= 0{,}5 - 0{,}3389$$
$$= 0{,}1611$$
$$= \underline{16{,}11\%}$$

Figure 4. A response with an incorrect p-value.

The response in Figure 4 shows that the teacher identified the p-value as –0.3389 instead of 0.3289.

Process-level responses in the probability layer There were some teachers (25) who read off a correct *p*-value but did not continue, while others interpreted the *p*-value incorrectly, as shown in Figure 5. The teachers in this group struggled to interpret the *p*-value that they identified from the table in terms of finding the probability that a person had a score less than 400, which required them to subtract the *p*-value from 0.5. These responses are indicative of a process-level engagement in the probability layer, showing that the *z*-score has been used correctly to identify the *p*-value.

$$P(x < 400) \qquad x = \text{score (3)}$$
$$Z = \frac{x - \mu}{\sigma}$$
$$= \frac{400 - 505}{111}$$
$$= -0{,}946 \approx -0{,}95$$
$$P(Z < 400) = -0{,}946, = 0{,}3289$$
$$= 0{,}5 + 0{,}3289$$
$$= \underline{0{,}829}$$

Figure 5. A response where the p-value was added to 0.5 instead of being subtracted.

Figure 5 illustrates a process-level engagement because although the person has correctly extracted the *p*-value and associated it with the *z*-score, he has been unable to reconcile that value with the position of the *z*-score and the required probability. Note too that the response does not include any visual or graphical image denoting the area that was required. Neither is there an image denoting the band of area that is associated with the *p*-value of 0.3289.

Object-level responses in the probability layer There were 79 responses (27%) where the probability of obtaining a score less than 400 was identified correctly. The standardization procedure was carried out correctly to produce a *z*-value of –0.495. Thereafter, the *p*-value associated with 0.495 was identified as 0.3289 and this was interpreted as representing the area under the curve between $z = 0$ and $z_0 = 0.495$. The required probability was given by 0.5 + 0.3289. Those teachers who arrived at an almost correct answer were also classified in this category. These responses were indicative of an object-level engagement in both the standardization and probability layers.

It is important to note that a response at this level required an object-level response at the standardization layer as well, which facilitated a view of *z*-scores as whole entities, separate from the process which produced them. At this stage the concepts at the two layers are coordinated to construct a process of identifying the area represented by the *p*-value. This process is then encapsulated so that it becomes possible to synthesize the *z*-score and *p*-values, to study their properties and to interpret the values geometrically in terms of area under the curve, allowing the teachers to generate the solution.

4.2. RESULTS FOR QUESTION 2

The responses to this question were also categorized in terms of the different levels of engagement with the concepts.

Pre-action-level responses in the standardization layer There were 78 teachers who produced a blank response and there were 38 who used an irrelevant method or incorrect algorithm. An example of an incorrect formula appears in Figure 6.

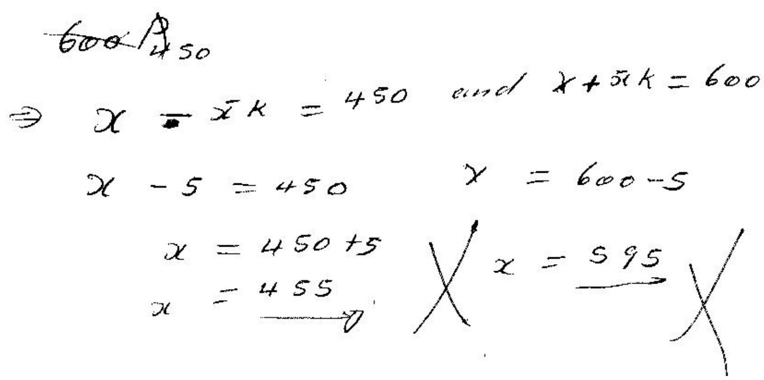


Figure 6. A response to Question 2 showing an arbitrary formula.

The response in Figure 6 shows the use of an incorrect formula. Another example in Figure 7 is a response to Question 2 which shows a very limited understanding of standardization.

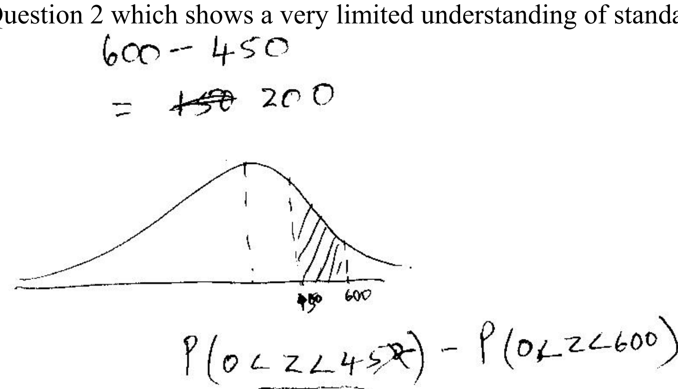


Figure 7. A response indicating a limited understanding of standardization.

Figure 7 shows that the teacher could not progress to the probability layer because his engagement at the standardization layer was very low, below even an action level. He was unable to convert the *x*-values into *z*-scores for both questions 1 and 2 and this prevented him from working out the associated p-values which were required in order to try to interpret the shaded portion on the graph, which he represented incorrectly in any case.

The 116 responses in this category did not show any indication that they had encountered these concepts previously, and are taken as being on a pre-action level.

Action-level responses in the standardization layer There were 18 responses where the standardization formula was applied incorrectly for Question 2, and these responses were taken as indicative of an action-level understanding of the standardization. The responses in this category were similar to the corresponding responses for Question 1.

Process-level responses in the standardization layer There were 54 responses where the corresponding z-values for one or both cases were calculated. However, no further progress was made in identifying the p-value. There were also responses such as the one shown in Figure 3 for Question 1 where the z-scores were taken as p-values.

Action-level responses in the probability layer Some responses (6) calculated one or two of the z-scores correctly but made errors of interpretation with respect to the p- values. These responses were taken as indicative of a process-level engagement in the standardization layer and at an action level in the probability layer. An example of a response where the correct z-scores were obtained but incorrect p-values were identified appears in Figure 8.

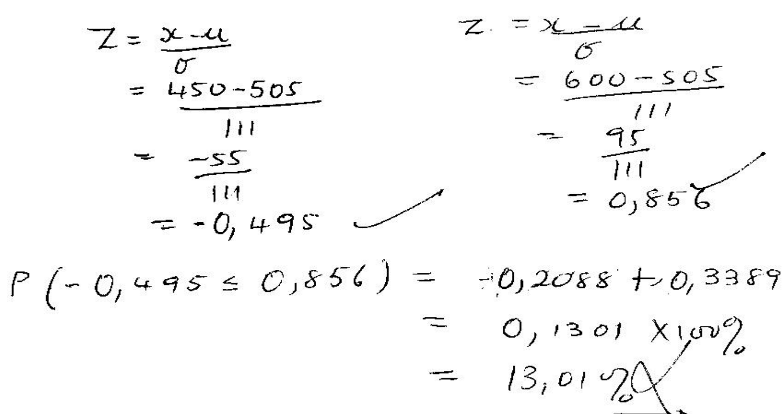


Figure 8. A response with correct z-scores, but incorrect p-values.

In Figure 8, the p-values are not correct. In addition the first p-value is taken as negative in response to the fact that the first z-score was negative. Perhaps the teacher tried to compensate for the fact that the z-score was negative by assuming that the probability was a negative value, which has no meaning.

Process-level responses in the probability layer There were 56 responses where the correct z-scores were calculated and the correct (or close to correct) corresponding p-values from the table were also produced. There were 45 responses where the p-value was read off correctly for both cases while 11 responses referred to only one of the two cases. Many of the teachers seemed to stop at that point, while others used wrong rules of combination for the two p-values. Some of the incorrect rules of combination that were evident were subtracting, taking a number between the two probabilities, subtracting each probability from 0.5 before adding or subtracting them, taking a combination of z-scores as the probability, and multiplying the two p-values. Even though these teachers did not progress further than obtaining the correct p-values, these 56 responses were taken as indicative of process-level engagement in the probability layer. Figure 9 provides an example of a teacher who subtracted the two p-values from 0.5.

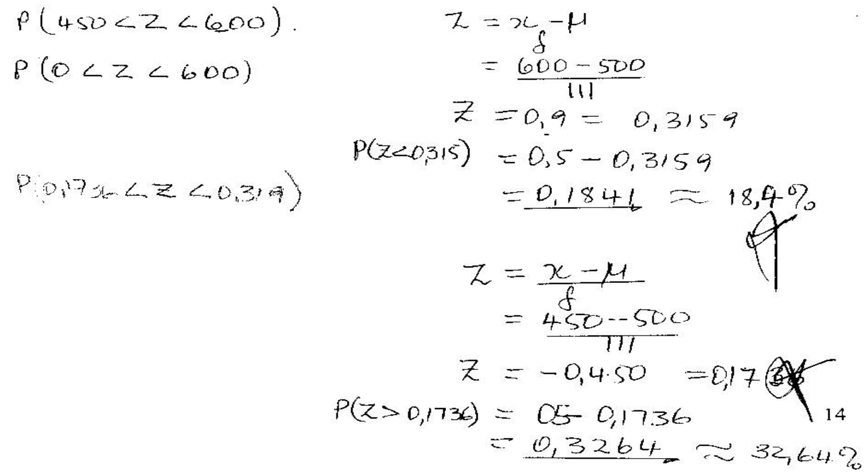


Figure 9. Response where the probability values were subtracted from 0.5.

In Figure 9, the teacher may have made a slip in substituting 500 instead of 505 leading to an incorrect *z*-score. She then read off the corresponding *p*-values. Her final step was to subtract each probability from 0.5, showing that she did not understand the representation of the *p*-value as the area under a specific portion of the curve. A further example of an incorrect combination of the two *p*-values is given in Figure 10.

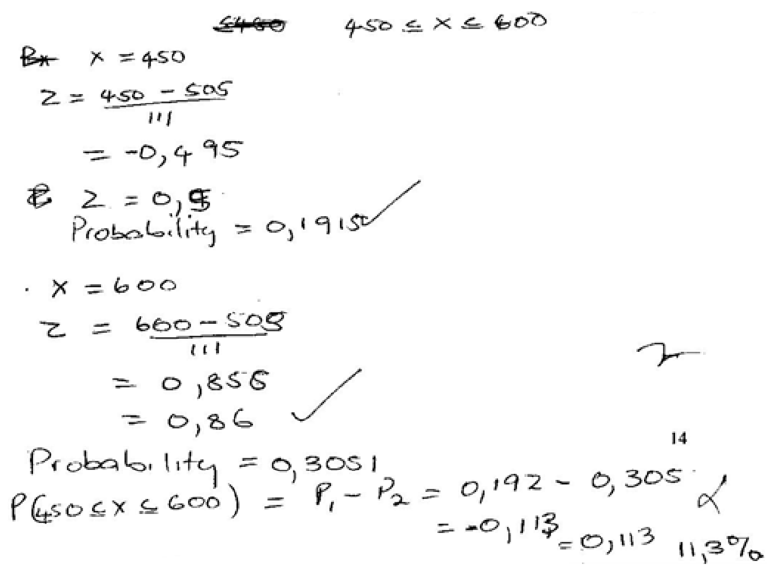


Figure 10. Response where the one p-value was subtracted from the other.

Figure 10 shows the response from a teacher who worked out the *z*-scores and then read off the correct *p*-values. However instead of adding the two probabilities, as indicated in Table 1, she subtracted them, indicating that she did not interpret the *p*-values associated with the *z*-scores 0.495 and 0.86, respectively, as representing the area covered by the curve between $z_0 = 0$ and $z_1 = 0.495$, and between $z_0 = 0$ and $z_2 = 0.86$, respectively. Such responses were taken as indicative of a process-level conception of the probability layer.

Object-level responses at the probability layer There were 40 responses where the *z*-values were identified and the corresponding *p*-values were found. These two p-values were then added to arrive at the correct (or close to correct) value. These 40 responses were taken as indicative of an object-level engagement at both layers. At this stage, the concepts at the two layers were coordinated to construct a process of identifying the area represented by the two *p*-values. This process was then encapsulated so that it became possible to compare the two *z*-scores and *p*-values to study their properties and to interpret the values geometrically as areas under the curve.

5. DISCUSSION

The results reveal that only 79 (27%) responses displayed an object-level engagement with the standardization process and reasoning about the probability in the unknown percentage problem using only one *x*-score. When the complexity of the problem was increased to considering two *x*-scores, the number of object-level responses dropped to 40 (14%), showing that the teachers had greater difficulty in comparing and visualizing the effect of two *z*-scores and their associated probability values. This means that from this sample only 27% of the teachers were able to solve the first problem while only 14% were successful at the second problem. The sample in this study consisted of mathematics teachers who are teaching mathematics (and therefore statistics) in Grades 10-12 in KwaZulu-Natal. It is a concern that such a small percentage of the sample displayed evidence of statistical reasoning in their responses to this task based on the normal distribution curve.

The value of using APOS theory as a framework is that it enabled an identification of specific difficulties as opposed to just registering the failure or success at the items. APOS theory asserts that the teachers are engaging at different levels with the concepts and therefore they would need different interventions in order to help them develop the necessary mental constructions. The study showed that there were large numbers of responses at the pre-action and action levels. There were 67 at the pre-action level for question 1, and the number increased to 116 for Question 2. In terms of APOS theory, these responses do not indicate any signs of engagement with the standardization procedure. These responses represent large numbers of teachers who have very little idea of what they are being asked to find. These teachers may need to be introduced to the concept of the normal distribution again, starting with the basic properties and then looking at simple calculations based on standardizing and 'unstandardizing'. They could then progress to problems of the type discussed in this study.

There were 54 responses in Question 1 and 24 responses in Question 2 which displayed an action-level conception at either of the layers. This suggests that the teachers who produced these responses have not been able to interiorize the procedure into a process and are relying on external prompts, hence they were unsure about how to consider the various values. Action-level engagement implies that the teachers have not had the opportunity to internalize the concept and need further engagement with the concepts or even repeated instruction in order to progress past this level.

The normal distribution curve is an important component of horizon knowledge (Ball et al., 2008) and its importance also lies in the fact that it is used to model many natural and psychological phenomena as stated by Batanero et al. (2004). These results indicate that mathematics teachers will need further professional development interventions which can help them understand this fundamental concept so that when they teach statistics they have an understanding of the field itself.

Many of the teachers' difficulties were located in the probability layer. In terms of the conceptions at the probability layer, there were two hurdles, one of which was the interpretation of the corresponding *p*-values as representing a proportion of area covered by the curve. Researchers (Cohen & Chechile, 1997; Reading & Canada, 2011) have emphasized that understanding the differences between a data distribution and a probability distribution is a key step in statistical reasoning. The results here confirm that many teachers struggled with properties of the standard normal distribution. A second barrier was the combination of the associated *p*-values in order to describe the proportion of area required by the question. Burrill and Biehler (2011) note that when students study probability with a formal approach, they will learn formalisms without understanding the phenomena described by this mathematics. In this case, the p-value given in the standard normal tables and the probability expressions $P(x < 400)$ and $P(450 < x < 600)$ can be seen as the formalisms. However, these formalisms are used to describe phenomena which in this case are the proportions of area under the standard normal distribution curve. The teachers struggled to translate the probability expressions into the corresponding area representation in terms of the distribution. Thus, the findings suggest that teachers need more time and experience in working with the concrete phenomena behind the formalisms.

The difficulties in the probability layer could also be linked to the demand associated with synthesizing two different representations. The cognitive difficulties of moving from interpreting a *z*-score that is represented linearly (along a horizontal axis) to a *p*-value represented by an area may be because these are different representations of the same situation. Duval (2006) asserts that two different types of transformations of semiotic representations can occur during any mathematical activity. The first type, called *treatments*, involve transformations from one semiotic representation to another within the same system or register (Duval, 2006, p. 110). The second type, called *conversions*, involve changing the system but conserving the reference to the same objects (Duval. 2006, p. 112). The movement from a *z*-score to a *p*-value may be seen as a conversion transformation, which Duval views as a cognitive threshold that is one of the main causes of learning difficulties in mathematics. It may be that the teachers need to understand the implications of the different representations. For example, they may benefit from activities which help them link the different representations – the *z*-score, the *p*-value and the area representation – across different normal distribution curves. Perhaps concrete data simulation activities may provide some help in this regard.

6. CONCLUSION

In this study, the responses of teachers to two related problems were analyzed in an attempt to understand the difficulties they experienced. The APOS framework enabled an identification of specific difficulties as opposed to just registering the failure or success at producing the expected answer. APOS theory suggests that the teachers are engaging at different levels with the concepts and therefore they would need different interventions or support to develop the necessary mental constructions. It is hoped that the use of the framework may encourage other statistics education researchers to explore the possibility of applying APOS theory to understand students' difficulties in other areas of statistics.

Many studies have identified low levels of mathematics content knowledge as a widespread problem amongst South African teachers. This study emphasizes the urgent need for mathematics teacher support, particularly in the area of statistics and particularly among those teachers who have only a three-year qualification from a teacher college. The study has shown that most of the teachers displayed limited statistical reasoning in the context of the normal distribution. Hence, it is important the teacher development agencies consider the results of this study and implement programs which can enhance mathematics teachers' understanding of statistical concepts.

One limitation in this program was the non-exposure of these rural teachers to computers. The teachers in this group were from various areas in the KwaZulu-Natal province and it was not possible to use computer-aided instruction, which is an integral part of most statistics education courses. Perhaps the absence of this resource may have limited the opportunities for visualization and engagement with the concepts encountered within the normal distribution. Experiences with simulated data and exposure to the many relevant applets may help such teachers experience the phenomena encompassed by the formalism, perhaps leading to a deeper understanding of the formalisms.

It is hoped that this study of teachers' engagement with a well-known problem (similar problems appear in many statistics courses) may have added to knowledge from which other researchers could benefit. Furthermore, by identifying areas of difficulties experienced by this sample, the results of some other studies could be juxtaposed with this in order to deepen our own understanding of students' engagement with statistical concepts.

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