WHAT'S MISSING IN TEACHING PROBABILITY AND STATISTICS: BUILDING COGNITIVE SCHEMA FOR UNDERSTANDING RANDOM PHENOMENA

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Teaching Probability and Statistics is more than teaching the mathematics. Historically, the math was first developed through analyzing popular games of chance such as involving the rolling of dice. This paper makes the case that the development of the understanding of Probability and Statistics is dependent upon building a "mature" understanding of common random phenomena such as the rolling of dice or the blind drawing of colored balls from an urn. An analysis of verbal protocols of 24 college students, who interact with and describe random phenomena involving the mixture of colored marbles, is presented, using cognitive schema to represent the subjects' expressed understanding. A cognitive schema representing a "mature" understanding of the random phenomenon is contrasted to a diversity of schema representing observed "immature" understanding. Teaching to explicitly build the mature cognitive schema is proposed.

INTRODUCTION: ON RANDOM PHENOMENA, PROBABILITY, AND STATISTICS

Historically, the mathematics of probability and statistics was first developed through analyzing popular games of chance such as involving the rolling of dice. The mathematics emerged 1650-1718, when mathematicians of those days (Pascal, Fermat, Huygens, James Bernoulli, de Moivre), through posing and solving problems related to dice games and other games of chance and gambling, first introduced the concepts of quantified expectation and probability, and related elementary mathematics (David, 1962). Given the pervasive presence in human culture of such random phenomena as the rolling of dice, over millennia, dating back to the times of the Greek and Roman civilizations, David (1962) and other authors on the history of probability and statistics (Hacking, 1975/2006; Gigerenzer et. al., 1989) have noted that it is remarkable how late in our history that the mathematics of probability and statistics emerged.

The late emergence of the mathematics provides an indication of its relative difficulty to understand. Consider, in contrast, that the origins of geometry date back to the Greek geometer Euclid (about 300 BCE), and origins of algebra date back to the Persian algebraist al-Khwarizmi (about 830 CE). For even the great thinkers of the past, the understanding of random phenomena that would lead to the emergence of probability and statistics was elusive. Thinking of such random phenomena in a new way was key to developing the mathematics. In this paper, I use the phrase *mature understanding of a random phenomenon* to refer to a view of a random phenomenon as reflecting or modeling concepts and principles of probability and statistics, that is, as having particular mathematical characteristics; for example (at the most basic level), seeing the rolling of a single die as having six possible outcomes, each with a probability of 1/6 of occurring on any trial.

Just as, for 17th century mathematicians, attaining a mature understanding of common random phenomena was instrumental to the development of the mathematics of probability and statistics; similarly, for students, attaining a mature understanding of common random phenomena is instrumental to the learning of probability and statistics. Understanding the math is accomplished through seeing how the math applies to concrete examples and circumstances. This is the principle behind the practice of using physical manipulatives to support students' learning of math concepts, e.g., using physical objects to illustrate arithmetic operations such as addition; and using squares and cubes to illustrate area and volume, respectively. For the math of probability and statistics, common random phenomena such as the rolling of dice or the blind drawing of colored balls from an urn are the counterpart to manipulatives to use to facilitate learning. However, the mathematical characteristics of these common random phenomena are not represented as directly within the phenomenon as is the case for the count of a set of physical objects or the measured dimensions of a geometric form. For example, for the rolling of a die, the mathematical characteristic of *probability* of an outcome is not a directly observed physical attribute of the phenomenon. Rather, the idea of quantified probability is indicated through the operation of the phenomenon over time,

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which has physical characteristics that contribute to the uncertainty and equal possibility of each of the possible outcomes; and whose outcome history unfolding over time, showing the frequency and sequencing of the possible outcomes, also reflects the uncertainty and equal probability of the possible outcomes. In summary, common random phenomena embody their mathematical characteristics (such as probability) indirectly and abstractly, contributing to the difficulty in seeing how random phenomena embody the math, and the difficulty of learning the math. A mature understanding of a random phenomena includes awareness of how the phenomena embody the math, and uses that understanding in describing and reasoning about the phenomena.

A COGNITIVE SCHEMA FOR MATURE UNDERSTANDING OF RANDOM PHENOMENA

A schema is a construct in cognitive psychology pertaining to the mental representation of conceptual knowledge (Minsky, 1975; Rumelhart, 1980). A schema organizes the characteristics or attributes associated with a concept into an integrated whole in memory, and is used in cognitive processing such as recall, recognition, reasoning, and decision-making. For example, a person's schema for "car" may include characteristics of appearance, speed, composite materials, maintenance needs, cost, how to start it, and so on. A schema (e.g., "car") may have subschema (e.g., "sedan" or "hybrid car"), hierarchically related to the more general schema.

In this paper, I apply the construct of schema to "random phenomena," as a means to formally describe a mature understanding of random phenomena; to illustrate the relative complexity and abstractness of the schema; to support analyzing students' understanding; to clarify teaching objectives regarding probability and statistics, and to identify directions for instructional improvement. Also, the schema provides a view of randomness that unifies "process" (mechanism) and "product" (outcome sequence) aspects, which have been presented as opposing perspectives in literature reviews over the years (Lopes, 1982; Nickerson, 2002).

A random phenomenon has:	
1. A physical mechanism, with a method to run repeated trials that each produce an outcome:	a. The *mechanism* has *features* that ensure no bias in favor of any particular outcome b. There is *a set of possible outcomes* for each trial, that set numbering more than one (=*n*); and each possible outcome has equivalent possibility, equal potential, equal *probability* (=*1*/*n*) to occur on each trial c. Outcomes on successive trials are independent, generated by the same mechanism, which is stable over time
2. Outcome sequences:	a. Over the long run (*m* trials), each of the possible outcomes has equal *expected frequency* in the outcome sequence (=*m/n*) b. There is *variation* in the frequency and pattern of occurrence of the possible outcomes among outcome sequences c. Over the long run, outcome sequences show no systematic order or pattern, and are usually mixed-up looking d. The probability of the next outcome in a sequence is independent of past outcomes, even when there has been an unusual sequence of outcomes such as a long streak of a single outcome category e. Orderly/ patterned sequences are possible to occur as *rare events*, since they are in the *set of all possible outcome sequences*
3. Predictability (by self/ others):	a. Don't know which outcome will occur, it could be any; so difficult to predict b. By chance, no matter which outcome one predicts, one has *probability* of prediction success =*1/n* c. Over the long run (*m* trials), *expected prediction success* is *m/n* times, or *1/n* of the time; and expect *variation* in prediction success across trials d. Long streaks of prediction success or failure are possible as rare events, since the events are in the set of all possible prediction results e. Particular prediction strategies are irrelevant to prediction success f. "Being lucky" is not a causal influence on prediction success

A schema representing a mature understanding of "a random phenomenon" appears above (in a relatively compact readable format). The schema organizes characteristics of a random phenomenon into categories: the physical mechanism producing outcomes, the outcome sequences, and the predictability of the outcomes. Related math is integrated into the schema, including probability, expected frequency, and variation. The schema, representing "mature" understanding (within a delimited/basic scope of knowledge), can be expanded to include additional mathematical knowledge, e.g., how to enumerate the sample space for sequences of *m* trials. Particular kinds of common random phenomena, including the rolling of a fair die, are subschema to this schema.

In the schema, the characteristics are interrelated and integrated into a whole, and readily available to apply in describing and reasoning about random phenomena. The schema is developed over time, not merely from being told, but from having experiences demonstrating the characteristics, that establish them firmly in mind. Some characteristics (1c, 2d, 2e, 3d, 3e, 3f) are included to suppress other naturally occurring ideas such as belief in luck, that are misconceptions from a mature perspective. The schema, by integrating information into a coherent whole centered around a real world phenomenon (that reflects more than just math), supports learning and retaining the math, and applying the math to reasoning about random phenomena in the real world.

STUDY DESCRIPTION

Subjects (Ss) were 24 undergraduates at the U. of Pennsylvania, aged 18-19 years old, 12 each male and female. Ss interacted with a Marble Shaking Machine involving the mixture of equal numbers of red(R), blue (B), and yellow(Y) marbles, depicted in Figure 1. Ss were presented with three identical random phenomena (three trays placed on the marble machine) and asked to play a game predicting successive outcomes with each. The mechanisms were chosen for their clearly unbiased natures (including sampling with replacement), so that Ss would be encouraged to interpret them as unbiased. Several checks were taken to verify that indeed Ss viewed the mechanisms as unbiased. In a balanced design, Ss were asked to use a different strategy for predicting outcomes with each phenomenon (tray), chosen from strategies that other Ss had used in previous experiments, namely, guessing a mixed-up looking sequence, a regular pattern, based on mechanism start state, or by daydreaming about the colors. For each tray, they kept a record of the color sequence to fall by inserting colored pegs into a pegboard, and kept track of their prediction success using plastic chips. Ss obtained an average success rate in prediction (4 out of 12) for two phenomena, and a high success rate (7 out of 12) for the other.

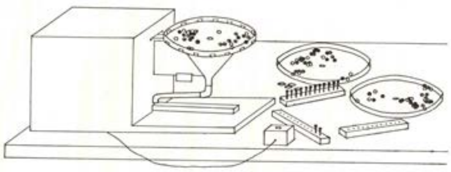

Figure 1. Marble Shaking Machine and Supplementary Apparatus.

Ss were asked to give a numerical prediction of their prediction success (out of 12 trials with a phenomenon) at various points in the procedure, and to explain why they predicted that number. After experience with the three phenomena (trays), Ss were asked to choose a tray to use for a final round in which they would be paid for each correct guess, and to tell which strategy they planned to use for that money-earning round and why. Of interest was how the high success using a particular strategy might influence Ss' judgments and reasoning regarding the phenomena. Near the end, Ss were asked what the word "random" meant to them, and whether the phenomena (trays) were random. A more detailed description of the experimental design and procedure appears in Kuzmak (1983). Results below are additional analysis beyond what is reported in Kuzmak (1983).

STUDY RESULTS ON UNDERSTANDING OF RANDOM PHENOMENA

Four of 24 Ss (17%) provided verbalizations that were consistent with the schema for mature understanding of random phenomena presented above. These Ss:

- gave a verbal indication that prediction success of 4 is expected by chance (see Schema 3c)
- predicted 4 for prediction success in the final round, despite having obtained high success with a strategy (see Schema 3d)
- in explaining strategy choice for the final round, gave a verbal indication that strategy makes no difference to prediction success (see Schema 3e)
- said that "yes" the trays/phenomena were random (4 Ss) or "appear pretty random" (1 S).

Twenty of 24 Ss (83%) provided verbalizations that were not consistent with the schema for mature understanding. The table below summarizes Ss' "immature" responses, highlighting ways that their understanding differs from mature understanding. Due to limited space, full schema descriptions are not provided for each category of immature understanding. The table illustrates that there is a diversity of immature understanding. Several Ss (25%) did not indicate that expected prediction success by chance is 4 or 1/3 of the time; their responses included predicting 5 "less than half" or 6 "an average number." Some believe prediction strategies are effective, while they agree that the phenomena are random. Some abandon the belief that the phenomena are random after having had an experience of high prediction success, not attributing the high success to chance.

Immature Understandings	**Number & Percentage of Subjects**	**Indicated 4 expected by chance (Y/N)**	**Predicted success for final round: Median (Range)**	**Are phenomena/ trays random? (Y/N/other)**
Strategy makes no difference to prediction success; expected success not calculated, but judged from experience	1 (4%)	N	4	Y
Gambler's fallacy, predicting mixed-up sequence works	5 (21%)	Y (all)	6 (5-6)	Y (all)
Able to predict without logical cause, ESP, intuition	4 (16%)	Y(3), N(1)	5 (5-6)	Y (all)
Possible mechanism-based way to predict	4 (16%)	Y (all)	5 (5-6)	Y (all)
Possible pattern-based way to predict, don't see why it works	2 (8%)	N (all)	6.5 (5-8)	Y (all)
Possible regular pattern-based way to predict (falling in twos)	1 (4%)	Y	5	"not conclusively"
Predicting regular pattern works (repeat BYR sequence)	1 (4%)	Y	7	"sort of"
Predicting based on mechanism works	2 (8%)	N (all)	7.5 (7-8)	N, "hoping ... not"

IMPLICATIONS FOR TEACHING PROBABILITY AND STATISTICS

Building up students' cognitive schema for a mature understanding of random phenomena is proposed to be adopted as an explicit teaching objective to facilitate learning of probability and statistics. Teaching then includes: explicit presentation of the schema for mature understanding, to facilitate seeing the integrated whole and the interrelationships among characteristics; and extended experiences interacting with common random phenomena, that demonstrate the characteristics and establish them firmly in mind, to then be readily available to apply to real world phenomena.

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