QUANTITATIVE REASONING AND CONCEPTUAL ANALYSIS AS A FRAMEWORK FOR TEACHING AND LEARNING PROBABILITY

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Thompson's theory of quantitative reasoning and von Glasersfeld's approach to conceptual analysis are underutilized tools in probability and statistics education. Both are valuable frameworks for researching how individuals conceptualize and reason about/with uncertainty as well as helping to inform instructional design around the same topics. We describe both conceptual analysis and the theory of quantitative reasoning and how they have shaped mathematics education. Further, we provide some instances where they have successfully been used in probability and statistics education. Sharing these useful tools from mathematics education has profound implications for the field given its tight linkages. Thus, presenting this framework has potential to provoke reflection within the field regarding what constitutes foundational probabilistic and statistical ideas and how instruction might support students' understanding of them.

BACKGROUND

How might probability and statistics educators support students at any level to build coherent and productive meanings for handling uncertainty? Further, how might we help students to find utility in conceptualizing probability in such a way as to help them deal with uncertainty? These difficult questions lie at the heart of probability and inferential statistics education. With the COVID-19 global pandemic, probability and risk took on a highly visible role in people's everyday lives. With this increased visibility, many of the challenges that probability and statistics education researchers have wrestled with for the past 50+ years were on display. Individuals struggled to make use of probability to make decisions, often resorting to heuristics.

We agree with Stigler (2016) that the concept of measuring uncertainty is a core component of any statistical reasoning. Both Weisberg (2014) and Hacking (2006) have detailed the rich and messy history behind the development of conceptualizations of probability. Within their works as well as the treatises on the probability concept by von Mises (1981) and Savage (1972), we find more than just historical notes and articulations of theory. We find an approach that can help answer the two questions we opened with: Thompson's theory of quantitative reasoning. Quantitative reasoning provides a natural and productive foundation for reasoning about uncertainty as well as the development of statistical and probabilistic reasoning. However, quantitative reasoning is not typically problematized nor an explicit focus in statistics and probability education. To this end, we will explain what this theory of quantitative reasoning entails along with the closely related tool of conceptual analysis (von Glasersfeld, 1995) and how they have helped to shape mathematics education. We will then discuss how these two tools can help us in probability and statistics education.

THOMPSON'S THEORY OF QUANTIATIVE REASONING (QR)

When we say "quantitative reasoning" we are specifically referring to the conceptual framework developed by Thompson (1994, 2011) meant to assist in describing and informing the learning and teaching of mathematics involving quantification from a cognitive perspective. Thompson built this framework out of constructivist-style conceptual analysis to describe and model an individual's mental operations. This creates a tight connection between the framework of quantitative reasoning (QR) and the tool of conceptual analysis.

Conceptual Analysis

Von Glasersfeld (1995) describes conceptual analysis (CA) as an approach for distilling how a person understands a particular concept into basic components of their mental actions. For von Glasersfeld, a concept is the dynamic mental re-presentations (mental acts of bringing past experiences

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to the fore of one's thinking) an individual has which have "been honed by repetition, standardized by interaction, and associated with a specific word" (von Glasersfeld, 1987, p. 219). A central premise to CA is that the analyst seeks to answer the question, "what mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (von Glasersfeld, 1995, p. 78). Given this question and that a researcher may only infer an individual's mental operations based on their observed behavior, CA is a method of model building. Thompson (2000) describes three usages of conceptual analysis: (a) to build second-order models for how another person might understand a particular idea; (b) to generate a model for meanings that should an individual have these meanings, then that individual is in a beneficial position for future learning; and (c) to generate a model of meanings that might inhibit the individual in generating an understanding of new situations and/or prevent the construction of new, more productive meanings. Thompson (2008) added an additional usage of CA: describing the "coherence of various ways of understanding a body of ideas" (p. 45). These four uses of CA tie to the goals of CA as well. Steffe's (1996) use of CA highlights students' mathematical realities (that differ from the researcher's) as valid and authentic constructions, and aspects of those realities that function effectively for them.

For interested readers, we point to the following examples where this style of CA was used within statistics and probability education: Saldanha's (2016) description of high school students conceptualizing unusualness, Saldanha and Thompson's (2002) work on students' images of sampling distributions, Liu and Thompson's (2007) investigation of teacher's understandings of probability, and Hatfield's (2018, 2019) descriptions of students' meanings for probability and their conceptualizations of stochastic processes. CA is a powerful tool for investigating how students might be reasoning. This power lies at the heart of the QR framework.

Quantitative Reasoning

To describe whether and how a person is reasoning quantitatively, the researcher must engage in CA. QR occurs when an individual conceptualizes a situation as consisting of a network of quantities and relationships between and amongst those quantities (Thompson, 1993). Within the QR framework, "quantity" takes on a specific meaning that goes beyond the commonplace meaning of number or amount. Specifically,

> Quantities are conceptual entities. They exist in people's conceptions of situations. A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability. A quantity is schematic: It is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality. (Thompson, 1994, p. 184)

The above quotation highlights an important tenet of the QR framework: we are modeling an individual's thinking/cognition. A quantity is a construct that a person (a student) conceptualizes in relation to objects within a particular situation. Further, their conceptualization of the situation is such that it supports them in thinking about objects imbued with features which they see as quantifiable. CA as we've described is the key tool that enables a researcher to investigate and make claims as to how a student might be conceptualizing a probabilistic situation quantitatively.

How a person conceptualizes a quantity occurs through acts of *quantification.* "Quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship […] with its unit" (Thompson, 2011, p. 37). There is an important back-and-forth that occurs between an individual's acts of quantification and their conceptualization of quantities. As a person engages in quantification, they will construct a quantity. As they work with and reason about and with that quantity, they may encounter shortcomings with their current conceptualization. This has the potential to occasion new waves of quantification as they attempt to resolve the issues they encountered (Thompson, 2012). This dialectic highlights that quantification is not a one-and-done process. Further quantification may take a significant amount of effort and time.

If we want students to conceptualize probability as a quantity, then we need to wrestle with the following questions:

- What is the object a student conceptualizes when conceiving of probability?
- What is the attribute of the object that the student conceptualizes as probability?
- What does a probability measure mean once we have such a measure?

These questions come directly from the QR framework. Perhaps most importantly, these questions and QR in general are not bound to any one philosophical position about measuring uncertainty. Answering these questions can be achieved through applying conceptual analysis, particularly by focusing on the meanings which would be beneficial for students to develop.

Throughout history, many scholars have engaged in QR and quantification. In settling what a measure of uncertainty means (along with how to get a measure and what is meant by measuring uncertainty), different scholars have taken different paths and arrived at their own meanings for probability. In the frequentist tradition championed by von Mises (1981), the object under study is a random process that can be repeated indefinitely to build up a collective of outcomes generated by the process. The attribute of this object is the propensity/tendency of the process to generate a certain set of outcomes, which von Mises conceived measuring through the limit of the long-run relative frequency. The meaning he gave to this measure was the percent of the time we anticipate seeing that set of outcomes. Such a framing makes probability a quantity. Suppose that instead of some process and collective, we concern ourselves with an act of gambling and the attribute of winning the bet. This sets the stage for the quantity known as "classical probability" or "chance," which Laplace interpreted as the ratio of the number of desired outcomes to total number of equally likely cases (Weisberg, 2014). De Finetti (1974) and Savage (1972) regarded probability as an amount of belief—an attribute that the person possesses. This led them to quantify "personalistic probability." In all three cases, individuals conceptualized some object, an attribute of that object, and what a measure of that attribute means. This highlights the power that QR has for probability education research. While the three philosophical positions are distinct, they all involve QR.

Impact on Mathematics Education

It is difficult to overstate the impacts of von Glasersfeld's approach to CA and Thompson's theory of QR. CA and radical constructivism led to a shift in mathematics education where researchers explicitly attended to the mathematical realities that students constructed, not just how the students performed. Further, CA has stimulated the development of several important methodologies in education research such as teaching experiments (P. Cobb, 2000; P. Cobb & Steffe, 1983; Steffe & Thompson, 2000) and design experiments (P. Cobb et al., 2003).

QR theory has spurred many research programs in mathematics education. Thompson's QR framework has been used to inform researchers' understanding of notions of transfer (Lobato & Siebert, 2002), trigonometry (Moore, 2012, 2014), rates of change (Johnson, 2012), exponential functions (Castillo-Garsow, 2012), calculus (Thompson et al., 2013) and covariational reasoning (Carlson et al., 2002; Zieffler & Garfield, 2009).

EXAMPLES OF APPLICATION IN PROBABILITY AND STATISTICS EDUCATION

As we've previously mentioned, Thompson's theory of QR and von Glasersfeld's CA are uncommon in statistics and probability education. To this end, we present two examples of applying these tools in the context of probability and statistics education.

Teacher's Conceptions of Probability

Liu and Thompson (2007) explicitly used CA in their investigations of eight US high school teachers' understandings of probability, all of whom had had prior course work and teaching experience in probability and statistics. Through the CA of teachers' statements and actions, they found that most teachers' understandings of probability were compartmentalized and not grounded in a conceptualization of distribution, and that their understandings depended upon how they conceptualized and understood the situation at hand. Liu and Thompson developed a framework for how a person might arrive at different probabilistic understandings. Their usage of CA helped them in several critical ways. First, by continuing to apply CA throughout the 8-week seminar they were able to create, test, and revise models of how those teachers were thinking about probabilistic situations. Second, their models helped them to think critically about how to support the teachers in shifting their thinking. For instance, Liu and Thompson (2007) framed a stochastic conception of an outcome's probability, x, as an individual's "expectation that the long-run repetition of the process that produced the outcome … will end with [that outcome] $100x\%$ of the time" (p. 121). They reported that while teachers started out largely with a non-stochastic conception of probability, throughout the experiment they began to conceive of more situations stochastically and, in several instances, to recognize that the same situation could be conceived both stochastically and non-stochastically. Finally, CA helped them design activities intended to elicit observable behaviors indicative of how the teachers were thinking. This also allowed them to create activities that tapped into the quantification that they teachers did and did not undertake. For several teachers, likelihood was not a quantity but a qualitative judgement of "more or less likely."

Relative Risk

The COVID-19 global pandemic provides an example of the quantitative reasoning framework's impact on research. The COViD-Taser project focuses on exploring how people interpreted media that contained quantitative data representations (www.covidtaser.com/). The project team routinely makes use of CA and the QR framework. In one study, their usage of CA suggested that citizens in both the United States and South Korea struggled to compare the risk of dying from the flu and from COVID-19, with only about 40% (13 of 32) making a multiplicative comparison of the risks (Yoon et al., 2021). They found that the participants struggled to reason quantitatively about individual risks, which became a barrier for making multiplicative comparisons. By using CA and the QR framework, they identified important ways of thinking for citizens to have a productive understanding of relative risks. In an attempt to help individuals reason more quantitively, the team created the Relative Risk Tool (English version: https://www.covidtaser.com/relativerisk; Korean version: https://www.covidtaser.com/relativerisk-ko). Joshua et al. (2022) detail lessons they developed for elementary, middle, and high school classes as well as their reflections in implementing such lessons. Their work highlights that reasoning quantitatively about probability/risk can happen at different ages and does not depend on algebraic formulas.

DISCUSSION AND IMPLICATIONS FOR THEORY AND PRACTICE

QR theory and CA can be powerful tools for both research and instruction. Saldanha, et al. (2022) use both QR and CA to not only provide a foundation for discussing the teaching and learning of statistics and probability, but also as a means for providing conceptual coherence. We want to stress the utility that these tools can bring to probability and statistics education, both in and out of the classroom.

As highlighted earlier, QR theory and CA have had significant impacts on mathematics education research. Statistics and probability education research can benefit from the usage of these tools. The QR framework provides a different perspective for researchers to work from. Diversifying our theoretical perspectives stands to increase the field's robustness and impact its development. The framework can directly influence the design of instructional activities for supporting students' understandings of statistical and probabilistic concepts and for providing a coherence across students' statistical and mathematical experiences. As another example, Hatfield (2019) provides CA not only of distribution, but of related concepts such as stochastic (random) processes, randomness, and probability. He further highlights how CA can be used to inform hypothetical learning progressions/trajectories (Lehrer et al., 2014; Simon, 1995). Research drawing upon both QR and CA stand on firmer theoretical ground and can point towards instructional design.

As guides for instruction, CA and QR can help guide educators in supporting students in developing coherent and productive ways of thinking. Further, QR and CA can help guide the construction of activities meant to support students in developing productive ways of probabilistic thinking. While calls have been made to reform how probability is taught in introductory statistics classes (G. W. Cobb, 2015; G. W. Cobb & Moore, 1997), textbooks either continue with the traditional approach or move the traditional approach into a supplementary section of the text. The QR framework is a powerful way to help students build productive meanings for probability in introductory courses. In the QR frequentist probability example, we see the elements that instruction should focus on to support students: conceptualizing the process, the notion of tendency, and a percentage of time (infinite repetition). Further, when formulas are necessary, QR provides an opportunity for students to have those formulas make sense to them quantitatively. That is, for students to see the symbols on a page not just as something to memorize in a particular configuration and/or as a call to calculate, but rather, as describing the relationships between quantities.

CA and QR as we have described here are powerful tools for education research. We hope that sharing these useful tools from mathematics education will have profound implications for probability/statistics education given their tight linkages. In presenting the QR framework and conceptual analysis, we aimed to provoke reflection regarding what constitutes foundational probabilistic and statistical ideas and how instruction might support students' understanding of them.

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