ASPECTS OF PROBABILISTIC THINKING AND REASONING WITH THE USE OF COMPUTER SIMULATION

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This work discusses the application of an activity as a didactic tool for approaching the concept of probability by confronting its classic and frequentist approaches. The activity takes place in a continuing education course for mathematics teachers who work in basic education. Research in the teaching-learning area of probability has highlighted the need for teachers to be better prepared for classroom work, with activities that can promote the development of probabilistic thinking and reasoning. In this study a qualitative case study research method is used. Among the conclusions, we found that the didactic situations we constructed allowed teachers to improve their knowledge, thus enabling them to recognize important elements for developing probabilistic thinking and reasoning.

INTRODUCTION

The 21st century world is complex and data driven. Informed citizens need to be data literate, and there is an urgent need to educate students and teachers to become literate. Preparing students and teachers for work requires that they be able to make decisions based on data, analyze data, and draw inferences and make predictions based on these analyses, which involves probabilistic knowledge.

Research on the teaching and learning of probability, such as the work by Batanero, Godino, and Roa (2004); Cavalcante (2021); Cazorla (2009); and Coutinho and Figueiredo (2020), has highlighted that teachers must be better prepared to teach probability, not only to use technology but also to propose activities involving probabilistic thinking and reasoning to promote students' development and assist in their decision making.

Therefore, the development of probabilistic thinking and reasoning today is crucial in teaching and learning elementary school mathematics. In Brazil, official documents such as the National Common Core Curriculum (BNCC) (Ministério da Educação, 2018) recommend including content related to probability. Further, in their "Probability and Statistics" axis, BNCC suggests that those objects of knowledge should be addressed from the early years of elementary school. BNCC also suggests that throughout basic education, the study of probability should be expanded and deepened through activities in which students perform random experiments and simulation and compare the results obtained by classical probability with those estimated by frequentist probability.

This study discusses the application of an activity as a didactic tool for approaching the concept of probability by confronting its classic and frequentist approaches. The activity is implemented in a continuing education course for mathematics teachers who work in basic education. The activity is based on a GeoGebra applet that simulates the Franc-Carreau (FC) game, which consists of tossing a coin that lands on a tiled floor with square-shaped tiles. The FC position occurs when the coin lands fully within only one tile. We further describe the game and offer some possibilities for dealing with aspects of probability using the game.

THE FRANC CARREAU GAME AND PROBABILITY

The applet used in the activity is freely available through the GeoGebra website at https://www.geogebra.org/m/zegKUvqP. Based on consecutive coin tosses in the game, one can build tables with accumulated frequencies and their respective probabilities for the coin landing in the FC position for these tosses. It is possible to toss the "coin" inside the "tile" and to toss this "coin" as many times as wished. Players bet on the whether the final position of the coin after it "lands," i.e., when the coin is stopped, is within only one of the squares, i.e., FC. Figure 1 shows a position of the coin where FC did not occur because the coin landed occupying part of two squares. Figure 1 also shows the game image after 94 moves, the number of occurrences of FC, and other data and commands that the applet offers on its main screen after each move of the simulation. The dimensions of the squares and the coin are fixed in the applet. The figure is composed of nine 5-cm squares, with a coin with a 1-cm radius.

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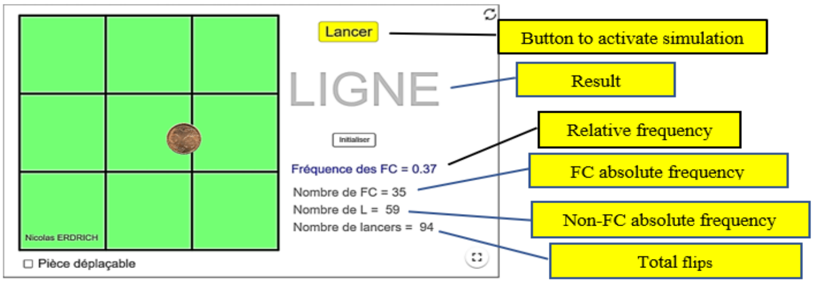

Figure 1. Data and commands available in the applet

From the frequentist probability perspective, one may ask how many flips would be enough to determine the probability that the coin lands in the FC position. We take the definition given by Ventsel (1962) in his work *Théorie des Probabilités*. The probability cannot be other than the number around which the values of the frequency series stabilize, $P(A) = \lim_{n\to\infty} Fr_n(A)$. The frequentist approach gives us a good estimate for this value. Comparing the probability calculation of the FC game by the frequentist approach with the calculation by the classical approach, we have P(A) = (numbers of successes / total number of cases). The latter is the definition proposed by Pascal and Fermat that was axiomatized by Laplace (2009) in his *Essai Philosophique sur les Probabilités* and that is still used in schools today.

METHODOLOGIC PATHS AND THEORETICAL BASIS

This study employed the qualitative research method of a case study, a modality that aims to know a well-defined entity in an educational system (Ponte, 2006). For this paper, we analyze results from one of the activities proposed to 17 basic education teachers in a continuing education online course. To develop the activity, we appealed to the theory of didactic situations to organize didactic situations to improve the teachers' probabilistic and pedagogical knowledge (Brousseau, 1986). The didactic situation dealt with the computer simulation of the FC game: simulate the coin tosses; write down the observed results; transfer the results to an Excel spreadsheet; and use Excel for other calculations. Then, the teachers were asked to construct the line graph that represented the sequence of results of observed frequencies for further analysis. The didactic situation used is drawn from Coutinho (2001) and displayed in Table 1.

The continuing education activity took place in a virtual environment using the Microsoft Teams platform, with channels made available for teachers to work on the issues to be solved in groups. Sometimes broader discussions were held in a large group, and sometimes they were organized in small groups. At each meeting, there was contextualization prior to the activity and the proposition of the didactic situation that should be developed in groups. At the end of the meeting, everyone returned to the large group to present and discuss what had been done in each group's channel.

This didactic situation lasted two class meetings of three hours each. All meetings were video recorded for this paper, and we analyzed the teachers' dialogues and written productions.

The environment in which the training took place with this activity can capture the nature of probabilistic thinking and reasoning that teachers mobilized in the face of their questions and offer them the possibility of expanding their repertoire. For our analysis, we took Borovcnik (2016) as the basis. Borovcnik synthesizes ideas about probabilistic thinking and how they are linked, including attending to elements such as randomness, the theoretical character of probability and independence, and risk, among others. Jones et al. (1999) pointed out elements for probabilistic reasoning and highlighted six key concepts: sample space, experimental probability of an event, theoretical probability of an event, probability comparisons, conditional probability, and independence. Based on these concepts, Jones et al established a classification of four levels of probabilistic reasoning: level 1 is associated with subjective or non-quantitative reasoning; level 2 is seen as a transition between subjective and naive quantitative reasoning; level 3 involves the use of informal quantitative reasoning; and level 4 incorporates numerical reasoning. We consider the levels of probabilistic literacy in alignment with the levels of reasoning and probabilistic thinking from Jones et al.

Table 1. Didactic Situation proposed to teachers of the continuing education course

a) Some initial procedures with the Franc-Carreau (FC) applet.
- I) Use the software to simulate coin tosses.
- II) Every ten throws, write down the number of throws that land in FC and the total number of tosses.
- III) Build a table in Excel with the results found for the absolute and cumulative frequency of the FC occurrence (create a column with the observed frequencies and, for each result, determine the cumulative relative frequency).
- IV) Do the same every 20 tosses and increase if necessary.

b) Analyze the items below.
- I) Could you find the stabilization of this relative frequency from the accumulated throws that resulted in the FC position? Explain your answer.
- II) How many repetitions did you perform in the experiment?
- III) What is the probability of tossing a coin and getting FC in this game?
- IV) How could you explain the calculation of this probability by the classical or frequentist approach?

c) Give your impressions about the activity.
- I) Did you have any difficulty in understanding and performing the simulation to “toss a coin and observe the final position after immobilization” using the FC applet? What difficulties?
- II) Did you consider the applet to be a good resource to work with students on the concepts of probability, simulation, and the classical and frequentist approaches to geometric probability? Explain your answer.

d) The relationship between the activity with the BNCC (Ministério da Educação, 2018) and Gal’s (2005) probabilistic literacy.
- I) What aspects of the BNCC (Ministério da Educação, 2018) would you be putting into action with the activity?

ANALYSIS AND RESULTS OF THE PROPOSED ACTIVITY

The teachers organized themselves freely into three working groups allocated in three Teams channels for the activity presented. We randomly named the channels A, B, and C for the analysis of teachers’ answers to the activities and their dialogues in each of these channels.

Although discussions among the three groups raised many common themes, we will present those of group C in this paper. The three groups began to toss the coin individually using the applet, i.e., each individual made a table and then compiled the results from group members as in Table 2. Group C observed the results and concluded that 20 throws would not be enough to estimate the desired probability. They continued to toss more coins individually until they reached 150 each. See Table 3.

Table 2. Result of 20 group C tosses

	FC	NFC	f
Teacher A	9	11	0.45
Teacher B	8	12	0.4
Teacher C	5	14	0.25
Teacher D	6	14	0.3
Teacher E	7	13	0.35
Total	35	65	

Table 3. Result of 150 group C tosses

	FC	NFC	f
Teacher A	68	82	0.45333
Teacher B	53	97	0.35333
Teacher C	48	102	0.32
Teacher D	50	100	0.33333
Teacher E	53	97	0.35333
Total	272	478	

Teachers began to recognize that a greater number of coin tosses was needed to determine the probability of CF occurring using the frequentist perspective, which indicates that, at this point, they were at the informal quantitative level of probabilistic reasoning as per Jones et al. (1999).

Nevertheless, they did not seem to recognize that experimental probability was determined from a large number of repetitions of that random experiment and that it was close to the theoretical probability (Laplacian) because they did not try to find such a way of calculating the probability. Nor did they provide any indication that they recognized the need to repeat the experiment many times. This would have classified them in level 4 of probabilistic reasoning according to Jones et al. (1999), the *numeric level.*

Group C identified that the results obtained in Table 2 were very different among group members, which did not lead them to conclude that such a simulation would reach a number that could be common to all, or at least close to it. At that moment, they seemed to identify the random character of probability through simulations, which according to Borovcnik (2016), is one of the pillars where probabilistic thinking resides.

When the group faced the number of simulations represented in Table 3, they formulated some questions in their discussion: How many simulations are needed to determine this probability? Will we really be able to arrive at a satisfactory number?

At this point, the teacher educators intervened and began to discuss the concept of frequentist probability in the three groups A, B, and C. This fact revealed that the teachers involved had not had any prior contact with frequentist probability or that they had not duly comprehended it.

Therefore, we can infer that the FC game activity was efficient in introducing the concept of frequentist probability and that it was effectively developed as an adidactic situation, in the terms of Brousseau (1986). Feedback given by the researchers responsible for the training and the dialectic of action, formulation, and validation were well characterized in teachers' dialogues and written productions, as illustrated in the text.

The development of the activity made these teachers search independently for a solution, mobilizing their knowledge and the knowledge introduced in the environment by the applet. They exchanged information about their individual experiences to make the experience part of the whole group, and they searched for a common result to carry out the validation. All of this occurred in an articulated way, effectively as a dialectic. Finally, the researchers responsible for the training institutionalized the concept of frequentist probability.

From the interference, the group changed their data collection strategy and began to divide tasks using screen sharing in the Teams channel. They showed the constructed table and only one of the applets tossing the coins. They generated another table in Excel itself, displayed in Table 4.

Table 4. Register of the results of 1,000 coin-tosses in the game presented by group C

Number of flips	FC	NFC	f	Number of flips	FC	NFC	f
50	18	32	0.36	550	193	357	0.35
100	36	64	0.36	600	212	388	0.35
150	56	94	0.37	650	234	416	0.36
200	74	126	0.37	700	253	448	0.36
250	96	154	0.38	750	271	479	0.36
300	118	182	0.39	800	291	509	0.36
350	141	209	0.4	850	311	539	0.37
400	156	244	0.39	900	334	566	0.37
450	156	294	0.35	950	354	596	0.37
500	182	318	0.36	1000	372	628	0.37

Then, the group began to entertain the possibility that the number of game simulations behaved such that the probability of FC occurrence approached a specific number. However, in the tabular representation of Table 4, they dispensed with the number of decimal places they worked with in Table 3 and admitted that the decision-making they had to do in the time allotted for the activity was not enough for them to formulate the answer accordingly. Because the proposed activity would require other questions to be answered, they decided to use results from only the applet, configured to present results with two decimal places, and then represented the graph with the data in the table to better visualize the data behavior, as displayed in Figure 2.

In the overall group presentations of results, they agreed that the graphic representation helped them to identify the probability of the FC position in the game when using the frequentist view. Yet, they found it difficult to calculate such a probability by the classical view so that they could compare them. Only one teacher could sketch his ideas about this calculation. Everyone provided evidence that they knew how to calculate probability from the classical point of view, i.e., the ratio between the number of elements in the event and the number of elements in the sample space. However, they did not recognize the area of squares as a possibility for determining these numbers. The geometric probability was not part of the repertoire of activities that these teachers would be able to propose to their basic education students. This fact allows us to hypothesize that there are other key constructs of probabilistic reasoning these teachers did not master, which opens up a perspective for research on levels of probabilistic reasoning following Jones et al. (1999).

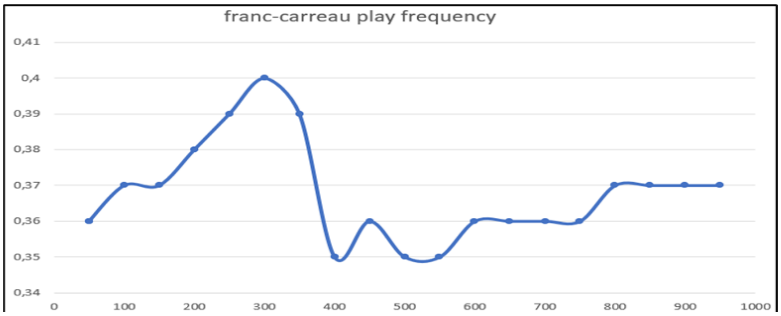

Figure 2. Graphical representation of the data in Table 4, presented by group C

The teachers did not have enough time to discuss all of the points related to the activity (Table 1), and they requested more time so that they could better understand all of the concepts involved in the activity and reflect on the activity more to deliver a new explanation in the large group session. After all teachers agreed, one week later, in a new Teams meeting, the groups presented not only the questions of the activity but some reflections on their practices as teachers. This fact leads us to what Jones et al. (1999) observe, that probabilistic reasoning develops over time. For these teachers, time for reflection was necessary to better develop their understandings of the concepts and ideas that emerged during the activity.

In this second meeting, the teachers presented new results of coin tosses in the game as tabular and graphical representations with more simulations. All identified there must be more throws to confirm the probability of occurrence of the FC event in the game. Besides those findings, one of the groups presented a calculation of the classical probability using not only the geometric explanation for it, but a generalization. They presented a formula that included the radius of the coin and the side length measures from the square, as displayed in Figure 3.

$$P(FR) = \frac{(l - 2 \cdot r)^2}{l^2} = \frac{(5 - 2 \cdot 1)^2}{5^2} = \frac{3^2}{5^2} = \frac{9}{25} = 0,36$$

Figure 3. Calculation of the probability of FC in the game, presented by group A

Regarding teachers' answers to the question, What aspects of the BNCC (Ministério da Educação, 2018) would they be putting into action with the activity?, the teachers showed a better understanding of indicators in the document after carrying out the activity, including conceptual objects of knowledge for basic education students to develop such as random events, the notion of chance, frequencies of occurrences, frequentist probability, and estimation of probability through the frequency of occurrence. Teachers' new understanding of document indicators seems to have created an opportunity for peer dialogue, which gives them new possibilities for working with basic education students using frequentist probability.

The three groups made suggestions for new possibilities of working with this activity. One of them reported the following: "In the classroom, we could use other geometric figures to perform the game, this would give more possibilities to relate the frequentist and classical probability without the applet with our basic education students, through concrete materials."

In this research, we could observe that the level of probabilistic literacy is dependent on probabilistic reasoning and thinking. Moreover, as probabilistic literacy increases, probabilistic reasoning and thinking become more refined.

CONCLUSION

The teachers participating in the course admitted that both the frequentist probability and the geometric probability involved new concepts. This fact caught our attention because, regarding their teaching time, the interquartile range allowed us to observe that 50% of teachers have 1.3 to 8.5 years of teaching. We hypothesized that they had had some contact with the probabilistic content to be addressed in the classroom. According to Huberman (1990; as cited in Bolivar, 2002, p. 54), this is the stabilization phase (four to six years of teaching) marked by the "consolidation of a repertoire of basic practical skills that bring security at work and professional identity."

However, the didactic situation, conceived of in light of the theory of didactic situations, allowed not only the approach of probability with teachers working in basic school, but also discussion on how these teachers could apply the game with their students. In this way, it allowed them to construct or deepen specific knowledge about probability and work on their own knowledge, which enabled them to recognize important elements for developing probabilistic thinking and reasoning.

Our findings indicate a need for continuing education courses for teachers of basic education that address probabilistic knowledge and its possibilities of articulating literacy, thinking, and probabilistic reasoning. Research is needed in this area to identify difficulties and new paths to follow.

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