SUBSTANCE OR STRUCTURE: HOW DO TEACHERS' CONCEPT IMAGES OF DISTRIBUTION COMPARE TO THOSE OF EXPERT STATISTICIANS

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Research into approaches to statistics education in recent decades has produced remarkably consistent recommendations, but this is not generally well reflected in the learning activities that take place in many classrooms. This multi-case study attempts to begin to understand why, by exploring concept images for the concept of statistical distribution. In this study of three teachers and three expert statisticians based in England, the concept images of the six participants were mapped through individual semi-structured interviews and then compared. The key differences identified between the expert statisticians and the teachers related to how conceptual elements were connected and navigated, with teachers relying heavily on their knowledge of assessment items for this. Additional similarities and differences are discussed along with some implications for professional development and further research.

INTRODUCTION

Research into statistics education is an increasingly well-trodden field, with recommendations around what constitutes effective teaching demonstrating a remarkable consistency (Ben-Zvi et al., 2018). Ideas expressed in the early 1990s have formed the bedrock of much of what has come since (e.g. Bargagliotti et al., 2020; Franklin, 2007), for example: emphasise statistical thinking; more data and concepts; less theory, fewer recipes; and foster active learning (Cobb, 1992). These recommendations have yet to find widespread prevalence within statistics classrooms at school level, despite notable progress in some countries, with many students still experiencing statistics education as algorithmic procedures applied to data by rote and teacher education programs rarely focusing on statistics content (Makar & Confrey, 2005; Schmid et al., 2014). Levy (2006) argues that "in their current form, traditional methods of teaching the pedagogy of data analysis and statistics fail to engage prospective teachers in examining their own knowledge for teaching" (Leavy, 2006, p. 107); this paper reports on a study designed to consider how this teacher knowledge of statistical concepts might be characterised and whether this is different to that of an 'expert' statistician, ending with suggestions for possible implications for professional development.

THE IMPORTANCE OF DISTRIBUTION

Core concepts in school level statistics include: informal inference; randomness; expectation; variation; and distribution (Watson et al., 2018). While these concepts are interconnected, this paper focuses on a single one of these concepts: distribution. The idea of a distribution of data underlies the uncertainty that distinguishes statistics problems from the more deterministic problems associated with the wider mathematics curriculum. According to Wild (2006), the notion of distribution underlies virtually all statistical ways of reasoning about variation and acts as a lens through which data can be viewed. As a consequence of this, it is important that teachers' pedagogical content knowledge (e.g. Ball et al., 2008; Shulman, 1987) incorporates a robust understanding of the concept of distribution. In its simplest form, the distribution of a variable can be described as "the values it takes and how often it takes those values" (Reading & Reid, 2006, p. 47). This description however hides the multiple layers of complexity arising from the way that aggregate properties give rise to identifiable structure and parameters which can then be used to make meaning from the data (Pfannkuch & Reading, 2006).

When considered from this aggregate perspective, one way of describing a distribution is through a framework consisting of five key elements: centre; spread; density; skewness; and outliers (Biehler et al., 2018). Furthermore, understanding distribution requires understanding not just these attributes of a distribution, but also many representations, classes, and inferential interpretations as described in Table 1 (Noll & Hancock, 2015).

Due to the central role of the concept of distribution in statistical literacy and reasoning, there exists a growing body of knowledge related to *reasoning about* distribution, which has been described as a "complex and challenging research topic" (Pfannkuch & Reading, 2006, p. 5). Much of the

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literature has been focused on characterising levels of understanding the concept of distribution from the perspectives of students and teachers (e.g. Bakker & Gravemeijer, 2004; Ciancetta, 2007; Jennifer Noll & J. Michael Shaughnessy, 2012; Reading & Canada, 2011), with a focus on how concepts of distribution are applied in statistical situations to solve problems.

Table 1. Components of the conceptual domain of distribution (Noll & Hancock, 2015, p. 367)

Representations	Attributes	Classes of distribution	Inferential Interpretations
Graphs (box plots, histograms, scatter plots, dot plots, pie charts etc.), tables, and spreadsheet formulas	Local features (individual data points, outliers) and global features (measures of centre, spread, shape).	Empirical versus theoretical, types of distributions (normal, uniform, binomial, chisquared, etc.), distribution of sample, distribution of population, and sampling distribution.	Signals in noise, p value, and empirical rule; variability within and between samples; and connections between probability models and statistics.

This paper reports a different approach to understanding how distributions are conceptualized in order to attempt to understand the level of complexity of teachers' conception of distribution and make comparisons to the ways in which experts may understand the same concept.

CONCEPT IMAGES

In this study the concept of distribution was considered from the perspective of the internal representations of the participants. Theories of internal representation arise from cognitive science and seek to describe human cognition and knowledge through mental structures (e.g. Augusto, 2014; Rasmussen, 1983; Sfard, 1991; Skemp, 1962). Tall and Vinner proposed the idea of the *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 2); this way of describing a concept was chosen because it was particularly aligned with the idea of distribution, a complex concept that consists of, but is not limited to, a collection of processes (e.g. calculations for average, spread); and images (e.g. types of graph, and common shapes that appear in data representations). While the formal description of a concept can be described as the *concept definition*, it is the concept image to which individuals resort when attempting to apply their knowledge – and this concept image may not be stable, consistent, complete, or particularly well-defined; reflecting the individual's current beliefs and subject to continuous change or revision (Greca & Moreira, 2000; Tall, 2013).

Given the multi-faceted nature of the concept of distribution its concept image is a complex system of related ideas, each of which also have their own individual concept image. For example, the arithmetic mean is an element of the concept of distribution, but is also a complex mathematical object in its own right. Models of teacher professional knowledge (e.g. Ball & Cohen, 1999) consider the role of both subject matter knowledge (e.g. curriculum content) and pedagogical content knowledge (e.g. approaches to teaching content). The concept image of distribution held by teachers encompasses both subject matter knowledge and pedagogical content knowledge, and this gives provokes a possibility that gaps in the concept image may be contributing to the apparent difficulty around embedding reform based in research recommendations for teaching statistics. In order to establish whether these gaps exist, and if so what they are, the following research questions were explored:

1) How can expert statisticians' concept images of the concept of statistical distributions be characterised?

2) How can teachers' concept images of the concept of statistical distributions be characterised?

METHOD

In order to explore these questions, a multiple-case study approach was used comprising three teachers and three expert statisticians all based in England. Of the teacher participants, two are secondary school teachers delivering a mixture of KS3, GCSE, and A-level teaching (age 11-18), while the third teaches A-level Mathematics and A-Level Further Mathematics (age 16-18) in a further education college. All the teacher participants therefore are engaged in teaching statistics content defined by the respective curricula that they teach. The three expert statisticians all work in UK Universities and the creation and dissemination of data, or application of statistical techniques make up a substantial part of their professional responsibilities. The participants all self-selected from within existing social and professional networks.

The data collection took the form of individual semi-structured interviews based around a series of six think-aloud tasks in which participants were provided with a simple prompt related to the concept of distribution and asked to “respond by describing the image that best matches your interpretation of the statistical context. For example, you may choose to draw pictures, write a description, invent data values and/or statistics, or some combination of these.” The individual prompts were selected to be accessible to primary school teachers as it was anticipated initially that at least one of the participants may be from a primary school teaching background. Each of the prompts had been used prior to the research study in professional development events as group discussion prompts and each was designed to align most naturally with different elements of the conceptual domain of distribution (Table 1); however, participants were encouraged to interpret them in the way that was most aligned with their own thinking. Examples of the prompts include “A dataset with a median of 25”; “The price of a bottle of wine”; and “The results of a test in which some students had studied for six months, while others had studied for one month”. A final prompt asked participants to “Write down all the words you associate with distributions of data and sketch any images you associate with distributions of data.” to capture anything that had not been explicitly covered in their prior responses, prompted by their thinking over the course of the interview.

For each prompt, participants were given some time to sketch and note their initial thoughts and provide real-time commentary. Then targeted follow up questions were given which challenged their initial responses in order to encourage participants to expand on their ideas and unpack their thinking further. The original intention was to conduct in-person interviews; however, necessary restrictions due to the Covid-19 pandemic meant that they were instead conducted online, with a recording made using online meeting software and transcribed later. This undoubtedly had an impact on the precise form of the responses as participants were required to sketch using a laptop on a virtual whiteboard rather than pencil and paper - the challenge of sketching in this way meant that the oral responses increased in their relative importance to the data analysis compared to what was anticipated.

A pragmatic approach was taken to interpreting the data, using a comparative method to generate simple coding based initially in the components of distribution exemplified in Table 1. Statements made by participants were taken at face value with minimal additional interpretation; so for example, mention of the mean was recorded, but no attempt was made to infer the participant’s precise conception of the mean unless they had explicitly provided additional information during the course of the interview. During the analysis of the six cases, additional themes emerged and were coded. The interviews were then reviewed again to ensure that they were applied consistently.

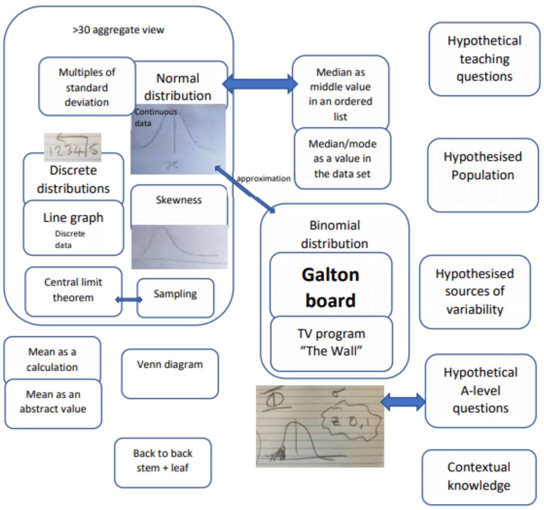

Figure 1. Example of a teacher’s concept image map

The set of codes were used to develop a ‘concept image map’ for each participant (Fig. 1) which recorded the conceptual elements of each participant’s concept image as well as connections between them and where possible, the relative importance or hierarchies between them.

RESULTS

All three expert statisticians appeared to have a concept image of distribution with a coherent structure. While the precise elements of their concept images differed, the individual elements were structured to produce a connected whole. This structure appeared largely dependent on their individual career as statisticians, with concept images structured around experimental design and modelling with data. The experts in the sample all had some teaching responsibility within their professional roles and this was evident in their answers, often code switching between describing their own interpretations, and which tools they would use to explain the prompt to others, for example:

Andrea: OK, well for me if I'm thinking about a data set with the median of 25, I'm probably thinking about a kind of box and whisker plot down there…But I always find that box and whisker plots can take a different degree of explaining to somebody.

Statements that acknowledged pedagogy were treated as a part of the participants’ concept image too, as they appeared to consider both their own understanding of the prompt and how they would teach it in tandem.

All three expert statisticians’ concept images contained broadly similar elements and showed evidence of both data-centric (focused on individual data) and aggregate (focused on emergent features and parameters of the data set) perspectives, which they could move between comfortably. Through their responses to the prompts, the expert statisticians demonstrated a rich web of connections between the various elements of their concept images with evidence of various different structures (Table 2). They used the context of the prompt to help navigate the concepts as they described them and in all cases, where context was not given they invented a context or contexts to fit the prompt and used this instead.

Table 2. Structural elements of experts’ concept images

Relationships between elements:	These were sometimes mediated by a third element, and sometimes simply unmediated associations.
Hierarchies of elements:	Some set of related elements for which an order existed in terms of how they were applied, or how important they appeared to be to the interview subject.
Groups of elements:	Some set of elements related to each other in a meaningful way. Sometimes this was an example of a set of elements that collectively made up a bigger concept.
Organising structure:	An element which was used to help navigate their concept image and identify relevant elements for a given prompt.

The teachers’ concept images contained broadly the same individual elements as the experts however there was little evidence of the same level of connectedness between them. There was evidence of both the data-centric and aggregate perspectives in teachers’ responses, but the teachers tended to remain focused on one or the other within their responses to each given prompt. Similarly the structural elements in Table 2 were evident but to a far lesser degree.

For the teachers, the taught curriculum was very evident in their answers, with many responses focused on how they would teach the content they associated with each prompt. Also evident was a *process view* of many of the elements – for example, how to calculate an average, or draw a particular graph. Perhaps the most striking element of all three teachers’ responses was their reliance on using hypothetical assessment items to help them make sense of and navigate the concepts that arose from each interview prompt. All three teachers frequently described GCSE or A-level style questions (these are usually short and highly structured assessment items) when explaining their thinking. It appeared that the teachers’ concepts of distribution were heavily influenced by the assessments for which they are preparing students and each had a bank of these hypothetical or part-remembered assessment items which they called on as a lens through which to view the individual elements of their concept images. Again, context was important for the teachers, but they appeared to use it very differently; as a link between the interview prompt and their hypothetical assessment items. In common across all six participants there were some recognisable structures and elements including: the idea of symmetry; shape (e.g. a bell curve) and context; aggregate and data-centric views; and multiple common representations (e.g. histogram, density curve).

CONCLUSION

While only a small exploratory study, some striking results were observed - in particular, the way in which the teachers in the study used hypothetical assessment items to structure and navigate their concept images. This has potential implications for pedagogy, and may begin to explain why reform of statistics education has been so challenging, with the idea of alignment between assessment and the goals of a ‘quality’ statistics education warranting further exploration and focus. Similarly, if recommendations for statistics teaching continue to advocate approaches based on statistical thinking and real data – more aligned with how the expert statisticians understand statistical concepts - it may be fruitful for future research to examine how experts think about statistical concepts with the goal of designing professional development programs for teachers that have the explicit goal of developing their concept images in line with those of experts. The study reported in this paper may support the approach taken in existing/historic professional development programs that encourage teachers to engage in ‘genuine’ statistical problem solving (e.g. Friel & Bright, 1998).

While provoking some interesting observations, significant limitations exist in this study, relating to the small number of participants, and self-selection meaning that only those with a degree of confidence and interest in statistics education were represented. Furthermore, the nature of the prompts meant at best an incomplete map of the participants’ concept images could be formed, arising from a set of questions designed to provoke consideration of a limited set of conceptual elements. These individual elements themselves were not explored in detail meaning that the concept image of, for example mean, held by the experts and the teachers may be substantively different, but is represented by the maps produced in this study as broadly the same. These limitations however do not detract from the key insight that teachers appear to structure their understanding of the concept of distribution through the lens of assessment, and the implications of this for pedagogy and professional development are potentially significant; however further research in this area is needed. This may be challenging, as any future research involving teachers must necessarily be conducted in an environment where high-stakes assessment is largely determined through policy drivers that researchers have little control over, but whose influence may be significant.

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