An historical exploration of the concept of average

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The term "average" is a common expression often used wrongly. The purpose of this paper is to look at the average or its equivalent through history and to examine how it is translated in schools today. First we will investigate the word itself, its origins, and its different meanings. We will see how the concept of average evolved. Average can be seen as a philosophical concept of "general" but its earlier uses probably developed within science. This concept also has some sociological significance in developing industrial societies where standardisation was essential. The concept of average played a role in politics: the idea of confidence intervals supports the statistical tools underlying opinion polls. The concept of average has become important enough to be taught in schools but little is said about the various senses of the concept and its difficulties are not really addressed.

Science underwent a period of considerable development in Europe in the nineteenth century, and this was reflected in the creation of new words to explain ideas and concepts, or issues which up to that point had been ignored. In medicine, new words derived from Latin or Greek were created. From the Middle Ages European universities had comprised four faculties: medicine, law, theologies and fine arts. Before attending the faculty of medicine, law or theology for their training in one of the three traditional liberal professions, students had to attend the faculty of fine arts (in France, enrolment in the faculty of arts took place in the secondary colleges such as those run by the Jesuits). And what would one study in the faculty of arts? Latin and Greek were traditionally considered as the doors of entry to real knowledge (Leon, 1967). The development of medical terminology in the 19th century was thus rooted in this cultural context, in common with all Western medicine.

In other scientific domains apart from medicine the influence of Latin and Greek was also evident, but rather less so. Developments were often expressed directly in the vernacular language. There was an absence of a long tradition. Eventually, scientific penetration of the universities often took place in reaction to the ancient culture of the humanities (Charle and Verger, 1994). While in certain circumstances, such as 19th century medicine, it was still seen as good taste to quibble over imperfect etymology and linguistic impurities (Moreau, 1972) elsewhere not much importance was attached to questions of terminology. Mathematics probably constituted the most extreme case of a break with the greco-latin tradition of the humanities. There was so little preoccupation with grammatical discussion that no new words were created to describe new concepts: these were taken from the living language. For proof, one can amuse oneself by leafing through a dictionary of mathematical terms: it is noticeable that terms existing in the current language abound, often with quite other meanings.

Using scholarly words certainly poses learning problems. The use of everyday or current words in a scholarly way also poses problems. Such words already carry other significations in everyday language, and in order to fulfill a scholarly function it may be necessary for them to be abandoned. Such an operation is doubly sensitive. In the first place, it is necessary to understand that the other senses are only abandoned during the learning period but remain adequate at other times. Secondly, other meanings can sometimes be in conflict with the sense which is being taught. So that we do not have to experience these difficulties, it will without doubt be necessary, before choosing this or that word, to follow the tradition of 19th century medicine, by creating another word not yet used by others. But history does not repeat itself!

We will illustrate the difficulties discussed by taking the example of the word *'moyenne'* used in French to describe what our English colleagues call the *'average'*' (or '*mean')*. Even if each language has its own style, the type of difficulties which we raise are universal and one finds certain similarities in other languages. The word *'moyenne'* derives from the indo-european root 'medhyo' in the sense of 'that which is in the middle' (Grandsaignes d'Hauterive, 1948). One finds a trace of this root in Sanskrit (madhyah) in Greek (meseuw) in Latin (medius) in Spanish (medio) in Italian (misaine) and in English (medal). In French, the word is primarily an adjective ('moyen, moyenne') and it reflects its etymology of 'situated in the middle'. It underwent a certain amount of distortion, becoming 'meiens' around 1120 and 'moienne' around 1360. By the second half of the 13th century the substantive form 'la moyenne' or 'average' was in existence.

Thus two meanings of the term have developed (Rey, 1992): the first corresponding to the initial meaning of the adjective—the average served to indicate or signify that which is equally distant from two extremes ('it is not large nor small, it is in the middle' - average) or the half of something ('have the average'). It was in this sense that it was used in arithmetic in the 17th century (the proportional average, *moyenne proportionnelle*, or simply *moyenne*, average) to speak about half of the sum of two numbers. In this case, the average is a middle point. Alternatively, another sense of the use of the word average appeared in the Middle Ages: the word served to evoke that which is most common, the most frequent, the most widespread or typical form (‘the average woman thinks that....’). The average represented the mass or majority, the modal value. When in the 16th century one talked of someone who was smaller in size than the average, one was indicating that they were a little different from the group.

These two last meanings of the use of the term are certainly related or close at least, as close as the median and the mode can be. But, in the 19th century, a third sense developed in statistics which defined the arbitrary quantity we know, and was given the name ‘average’; thus, the expression ‘on average’. Again, physics itself already talked about ‘average speed’ defined as the relationship between a distance travelled and the time necessary to cover it, where the expression ‘on average’ is used elliptically in this instance for ‘average speed’. The use of the word ‘average’ in the expression ‘average speed’ came from astronomy: the average movement of a star corresponded to the uniform movement which would be needed to travel its trajectory in the same time as its actual movement. One can imagine the confusion (Diderot, 1778). What links can be made by a student when he or she is told that a car is going at an average of 100 km per hour, and that the class marks are an average of 8.5 over 10? The confusion is made even greater by the expression ‘on average’ in the current sense as something approximate. One says, for example, ‘that makes an average’ simply to say roughly that it is equivalent.

The arithmetical average was not widely used until the 19th century. The concept is, nevertheless, much older. In antiquity they had already been aware for several centuries of variations in the movement of the sun, the moon and the planets. The Babylonions (from the 5th to the 3rd century BC) attempted to make many observations of the same phenomena to establish certain parameters, such as their position at particular times of the year. It is not known, nevertheless, if the idea of the average was used then. Among the Greeks, who left written records, the technique of estimating the position from the centre of observations (by taking half the range of the observations) was employed by Hipparchus in the 2nd century BC. The works of Ptolomy suggest that the average was still not a technique in general use. It is only with the work of Tycho Brahe (at the end of the 16th century) that the use of the term average becomes clearly distinct, in the sense that it is used to eliminate systematic errors. In the 18th century, scholars were adept at the use of astronomy. In the 19th century statistics developed as one of the humanities: one can make a broad analogy between, on the one hand, human and social reality, and on the other, physical reality. The development of statistics occurred in these two fields which are, therefore, interdependent. The idea of the average which was current in astronomy, now became a concept used in the social sciences (average man, average classes).

It could be argued that these terminological difficulties are of little importance, that the concept of the average will become clear some day or another, and that it will have its own meaning comprehensible to everyone. One certainly hopes that the academic system contributes to this. In effect, languages evolve just as society does. It will be interesting to investigate what other new realities will help to define the use of the average. It is known that an interest in the average occurs in the areas of science, commerce, industry and politics. But is it sufficient for a language to permeate into current usage? There are words, like some of those used in medicine, whose meanings are only learnt within a specific context, and as part of a specialised lexicon. One could ask why this is not also the case with the use of the average.

It strikes us, nevertheless, that there is in the modern world a new reality which has never had an equivalent in the past. To appreciate this we can take the example of the introduction of the metric system in France. This is a good example of a scholarly event which, through social necessity, became an event involving everyone. Before the French Revolution, not only was there a proliferation of weights and measures, but there was greater disagreement over measurement. Measurements could be a little less, just right or in excess. The question became a social issue: peasants complained that their masters used their own measurements and to their own advantage. The installation of a new system of measurement, unique and equal for everyone, was one of the first actions adopted by the Revolutionary government, and thus introduced equality between citizens by abolishing privilege. We know the rest of the story: scholars set about constructing a system of measurement, which they wanted in their *naïveté* to take from nature itself, before being adopted and utilized by all nations (Hocquet, 1995). The school system in its turn began applying the metric system and today it is in daily use throughout the world.

To return to the concept of the average—this has been more or less useful for the ordinary citizen for at most the past 50 years. Merchants and consumers again adopted secular customs. Goods were purchased most often in bulk, small markets were common and mass production rare. The optimal use of machines, modes of transport, stock-keeping, price reduction, all such things did not form part of the preoccupation of the community (Lafranc, 1941). Now, everything has changed. Commerce has become globalized. Products are bought by the consumer wrapped, weighed and standardized. The individual no longer considers him or herself alone in the world: it is clearly visible that there are millions of consumers like them. Modes of communication have changed our imagination completely. The consequence? The consumer is now confronted with large quantities and a greater variety of products from which they must choose. It is necessary to face a problem which has never before been confronted, large quantities and variety of products, where there is a need for the consumer to appreciate, to make up his mind. The consumer needs tools to be able to do this, just as the French citizen needed a means of breaking the ties with the ancient feudal system. In retrospect, it should not astonish us, if in secondary school teaching, we have chosen to introduce a notion such as that of the average.

It is not surprising that the first mention of the average that we found in *"Quick at Figures" designed especially for Commercial Schools and Business Men* (Roy, 1892) was reserved for business classes. In fact the author talks about "alligation" saying it is a simple application of the average and gives as an example a problem of weighted average. It is interesting to see that to solve an inverse situation, where the average and the price of the ingredients are given and the quantities of each ingredient are unknown, the author uses an arithmetic method base on the difference to the mean. In the first part of the 20th century, the idea of average was introduced in middle school (the equivalent of high-school) as a problem of price by unity, hence, a simple division (FEC, 1905). The first mention of average of marks appears in 1935 in a French textbook for elementary schools. From then on, arithmetic average is presented as early as the fourth grade in elementary schools. The contexts vary, we see problems of distance, food, repairs and baseball! In the same period in textbooks (Canadian and American) we see a pedagogical approach. First the sum and divide method is explained in a context, number of repairs by repairmen, for example, then there is an attempt to give a meaning to the average: "…it means that if they made 317 repairs all together and if all five had made the same number of repairs, each one would have made $63\frac{2}{5}$ repairs" (Buswell, Brownell, John, 1938: 37). The data are sometime presented as a list but most of the time, there are still only one, two or three values.

Although, the presentation of the addition and division "method" dominates textbook presentations, occasionally we find some interesting approaches. In *Algebra for*

Problem Solving (Freilich, Berman, and Johnson,1957), the authors suggest starting with an estimate average, and go on working with the deviations to the assumed average, finding the average of these deviations and adding it to the estimate average to find the correct answer. It is true that this was before the introduction of calculators which took out the toil of computing big numbers which also eliminated some interesting approaches. Although it is presented as a method, working with an assumed average could develop an deeper understanding of the average. Later on, variations in the final question are found. In *Calcul Nouveau* (Colas, 1963) a textbook for 5th grades, not only are students asked to compute the average, but also to find a missing value, the average given, to find the total of the data with the average and the number of values. This is interesting because in elementary school, the students do not yet know any algebra and have to rely on arithmetic.

Finally, the average is introduced as a statistic amongst other values of central tendency measures in a chapter titled: *Introduction à la statistique* in a 8th grade textbook (Ouellette and Desroches, 1969). This brings up comparison between mode, median, and average, and enriches the concept. Is in the seventies that statistics, including average, becomes a true part of the mathematics curriculum. At the same time we see many other changes. Data are now presented as a list of many observations, contexts are various and include more socio-economic variables, numbers are larger but the questions tend to be stereotypic: "Find the average value". The difficulty is created by the difference in the presentation of the data: list, grouped data, or tables, etc… The solutions proposed are algorithmic and more and more symbolism is introduced: $\bar{x} = \frac{\sum_{1}^{n} x_i}{n}$ (Carter *and al.*, 1980). The introduction of calculators also induces a more algorithmic approach.

Exceptionally there is an attempt to confer a meaning to the average using a representation. *Chantiers mathématiques* (Lemyre, 1981) illustrates piles of books and talks about distributing them evenly to find the average. However, most of the time the attempt at representation is limited to a title under the figure of a fulcrum, mentioning that the average is the centre of gravity (Assouline *et al.*, 1995) without further explanation.

As we have seen, the concept of the average was introduced into schools in a commercial context. Mixture and other problems are treated arithmetically. Computing methods use mostly two characteristics of the mean: equal distribution and sum of deviations. However, this is not done explicitly. Only in the middle of this century do we observe, in elementary school textbooks, attempts at verbal explanation aiming to give a sense to the computation of the average. With the seventies, the average becomes part of a statistics chapter but in some ways loses some of the interesting applications that had previously been linked to the concept .

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