TWO DICE, TWO SAMPLE SPACES

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To justify the probabilities for the outcomes of dice games, learners construct informal sample spaces. For two distinguishable dice, one sample space, for the student work reported here, is the usual product space, denoted A, with 36 equally likely points. When the dice are indistinguishable, the observed (rather than theoretical) outcomes were represented by a sample space B with 21 points, onto which A maps, by a map which is generally, but not always, two-to-one. Here we study five preservice teachers, who work through, informally but rigorously, the relations between A and B, and then the probabilities of events modeled by subsets of B, in effect by pulling these events back to corresponding events in A. Both spaces seem necessary, as their connections need to be explored in detail by these learners.

It is widely understood that the investigation of games and strategies helps learners to build theories, based on compelling personal experience, forming images and intuitions which become the subjects (Dann, Pantozzi, and Steencken, 1995) of explorations which lay stress (Maher, Martino and Pantozzi, 1995) on careful listening and questioning. The tasks we study here (Maher, 1995) proceed as follows (1).

Game 1: Two players roll one die. Player A gets a point if 1, 2, 3, or 4 appears. Player B gets a point if 5 or 6 appears**.** The first player to receive 10 points wins.

Game 2: Two players roll a pair of identical dice. Player A gets a point if the sum of the two numbers on the dice is 2, 3, 4, 10, 11, or 12. Player B gets a point if the sum is 5, 6, 7, 8, or 9. The first player to receive 10 points wins.

In each case, the learners are asked whether or not the game is fair, and why they think so. Further, if they think the game is not fair, the learners are invited to suggest how one might modify the game so as to make it fair.

The players here are five preservice teachers, in the second semester of an experimental version of the mathematics course for elementary education majors (Mathematics 305-306) at Brigham Young University, taught by the authors in the form of a teaching experiment (Speiser and Walter, 1998) to investigate how education students build important mathematics, through task investigations, for themselves. The work we study here, by the focus group in Speiser's section, took place on February 7, 1997, a month into the second semester. After working on the games themselves, our students watched videos of the Kenilworth sixth-graders (Maher, 1995) engaged in similar investigations, and then discussed the children's work from the perspective of their own experience. This process, personal investigation first, then careful study of young children as they work on similar tasks, defines the focus and direction of our course design. By the second semester, a classroom culture, centered on energetic questioning, experiment, and detailed verification, had firmly emerged.

The student work reported here (on Game 2 above) describes some very thoughtful explorations by elementary education students. We begin with a discussion of the underlying mathematics, together with some cognitive questions which that mathematics raises, based in large measure on discussions of the Kenilworth children's investigations, which our students' work will closely parallel, and quite likely help explain.

THEORETICAL FRAMEWORK, RESEARCH QUESTIONS

The sample space for rolling two identical dice is not uniquely determined, but it is fairly narrowly constrained. How do learners work with its constraints? In practice, we have seen children construct either a sample space, which I'll denote by A, with 36 outcomes, or else a smaller sample space, which I'll denote by B, with 21 outcomes. (To be precise, the space A can be constructed as the product of the set {1,2, ... ,6} with itself. In a similar style, we can define the second space B as the quotient of A, modulo the symmetry which exchanges points (x,y) and (y,x) of A.) In the model given by A, the two dice retain their "rugged individuality" even though we cannot distinguish them in practice, while in the second sample space, B, visually indistinguishable outcomes (such as (2, 1) and (1, 2)) have been identified.

Once given a sample space, however, the probabilities are uniquely determined by a rigorous process. How do children see this process? One pattern we see frequently is an argument that the 36 sample points in A are equally likely, followed by the construction of the natural mapping of A onto B, which determines the probability of an element, say b, of B to be n(b)/36, where n(b) denotes the number of elements of A which map to the given element b. (Here n(b) is either 1 or 2.) Direct inspection of the map from A to B tends to settle arguments about whether or not the 21 outcomes given by B are equally likely.

The really interesting research questions here would seem to be: What, exactly, are the doubts and disagreements which emerge among the learners here? How, precisely, are these resolved? What kinds of theories are the learners building as they do so? What kinds of evidence seem most compelling, especially for learners, here? And why?

For example, consider the arguments which frequently surround the 36-element sample space A. On the one hand, at the level of the Rutgers tasks themselves, there are three competing models to compare: two identical dice thrown together, vs. two differently colored dice thrown together, vs. one die thrown twice. On the other hand, at the level of children's disagreements: the potent observation that identical dice can't be distinguished, which militates toward the sample space B. This argument cannot be dismissed, due to its evident empirical validity (informal experimental physics: what can we *actually* count and measure?) so the assignment of probabilities to points of B becomes important *theoretically* as well as practically.

Data

Rachael builds the sample space B to model the possibilities for two indistingishable dice thrown together. Shauna and Jill build the sample space A to model the possibilities for a white die and a green die thrown together. Intense discussion leads Rachael to agree, with everyone *supporting* her decision to maintain B as her sample space, that the map from A to B assigns unequal probabilities to the points of B, with "doubles" counting less. Jill and Shauna's argument, essentially that coloring the dice doesn't change their numerical behavior, will convince Rachael only after the map from A to B has been inspected. This excerpt, early in the group's discussion, illustrates how carefully the difference between A and B must be examined.

Rachael: OK, then. Let me tell you what I'm saying.
Jill and Shauna: OK.
Rachael: I'm wondering... I don't know if I'm right. I don't even think I'm right, but I don't know. If this [points to (1,2) in Shauna's chart] has one chance, and if this [points to (2,1)] has one chance, because they each have 50-50 chances of happening. Right? OK,OK. Wait, wait wait. [Takes two dice from the table.] Let me do it with this. One and two... and two and one... one and two and then two and one have the same chance of happening, right? OK. But... so

	together are they just one chance, or are they two different chances?
Shauna	[with one white and one green die beside her chart]: They're two different chances of getting...
Rachael:	But when you roll them at the same time...
Jill:	It's still two different chances.
Shauna:	But, see, if we... you know... if we took... but wait, but listen... we said that these are all the combinations of rolls you can roll, and we divided by... it would give each one's percentage of happening... that one [points to the row listing (2,1) and (1,2) on her chart] has got to be double the chance of happening [points to the row listing (1,1) alone] of this one.
Rachael:	OK.
Shauna:	Do you see what I'm saying?
Rachael:	I do. But I'm wondering if it's true.
Shauna:	It has to be, so therefore... therefore you have a pretty good chance of rolling three.
Jill:	If you count these all separately and add them up like you wanted to do, two and one, and then it's one and two…there are 36 of them... This one has got... Say their percentage was 10%, then that [points to (1,2)] has 10% and [points to (2,1)] that's got 10%, and this one [indicates both] has a total of 20%, and that [points to (1,1)] has only 10%.
Rachael:	OK. But I'm wondering if together... if we roll them together, does it... is it really different?
Shauna:	Well, listen. If this... if, I'm saying, this has a greater chance... Think of it logically: if there are all these kinds of ways [points to her chart] you can get a 7, a sum of 7, wouldn't you think it'd have a much greater chance?
Rachael:	That, yeah.
Shauna:	Since there's just 7, right?
Rachael:	Oh, yeah. But, also it has a much greater percent chance because there's one and six, two and five, three and four, and this [points to (1,1)] has just one.
Shauna:	Exactly.
Rachael:	But if you don't count these [indicating the corresponding duplicates] it still has a greater chance.
Margie:	That's what [indicates her notebook] I'm saying here.
Rachael:	Yeah. But this one... I don't know. I see what you're saying. Maybe you're right.
Margie:	I think it's just because, twelve, there's only one way to get it; two, there's only one way to get it; three there's only two ways...
Jill:	Look at these [points to her notes]. These have a whole bunch of ways...

DISCUSSION

These learners' representations take the form of lists and charts. Shauna and Rachael list their possibilities in horizontal rows, for the sample spaces A and B respectively. Holly, who follows their discussion closely, lists the possibilities for each given sum vertically, by blocks, for the space A. The argue by referring to their charts.

What convinces Rachael, later in the discussion, is surely not the precise *numerical* values which the calculations based on A, as Jill explains them, will eventually predict for points of B. Rather it is the careful counting of the number n(b) for each point b of B.

The numerical calculation of probabilities, in fact will come last for all five members of the group, despite a probe from our assistant, because the key points to be established here are *theoretical*: the spaces A and B, each modeling a *different* game; the map from A to B, and its relation to both games; the determination of n(b), by inspection of the map; and finally the basic observation that the 36 points of A have equal probability. From these theoretical conclusions, which our students argue carefully, the probabilities needed to complete the dice game task follow immediately, in such a way that everyone is right.

Our students, completing their investigation, drew several conclusions of their own.

Rachael:	I'm sad that doubles are [less likely].
Shauna:	Really.
Jill:	Your whole life goes ...
Margie:	That's why its so hard to get snake eyes.
Rachael:	Snake eyes in the back of your head.
Margie:	That's why we don't gamble.
Rachael:	But it would be just as hard to get, like, six, six.
Holly:	So that's cool. Like, every time , every time I play Monopoly now, I'm gonna, like, the probability of ...
Rachael:	The value of snake eyes.
Margie:	Grrrr! I got a one thirty-sixth chance of getting these double ones!

It is important to observe that the sample spaces A and B were used to model *different games*. Indeed, B models the game with identical dice, which is precisely the game proposed in the students' task, while A models an analogous game, but with *distinguishable* dice, as Shauna may have emphasized by placing a white die and a green die conspicuously beside her chart of possibilities. The second game, whose 36 outcomes are equally likely, was needed to compute the *unequal* probabilities of points of B. This observation strongly supports our conclusion that both sample spaces, and the map between them, seem necessary, in order to explain solutions of the dice game task.

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Notes

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