A Complementarity Between Intuitions and Mathematics

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1. Learning as a dynamic interplay between intuitions and theory

It is one of the strengths of mathematics that it allows one to concentrate on structural properties, so that one can make generalisations of a specific context: one need not worry about all the puzzling peculiarities of real situations. Concepts used within this mathematical theory get their meaning indirectly by the relations codified within the axioms and the theorems that can be proved on their basis. Within this structural approach, probability is a bounded measure (like area) and the additional concept of independence plays a key role which is indicated in such theorem groups as the Law of Large Numbers or the Central Limit Theorem. The logical body of a theory should justify concepts in their specific appearances and should make ideas communicable.

However, there is some sort of imagination which seems to act as a driving force in the development of mathematical theory, before concepts are clarified, or while they await construction. Mathematicians do not like to speak about this world of intuitive images that they had in their mind when searching for a theory. These intuitive ideas are only partially uncovered in a mathematical presentation. It is necessary to codify ideas by using abstract concepts, otherwise the ideas are too vague to be communicated. On the other hand, the abstract concepts seem to be not understandable without sharing similar images. The theory cannot be "decoded" without sharing some of the intuitive ideas in the background. This common sharing of intuitive ideas distinguishes scientific communities, as can also be seen by the big efforts it takes young researchers to get into this community: progress is very slow at the beginning, but speeds up once one has access to the common pool of intuitive images.

There seems to be an irrevocable link between intuitions (intuitive ideas, intuitive conceptions) and theory (abstract models, concepts). It is not possible to separate these two aspects, each of which is necessary to understand the other. For such aspects, not really separable without loss of genuine meaning, it has become popular to say they are complementary (this term stems from Bohr's views on modern physics and has been introduced to didactics by Otte). We shall continue to use the term "intuition" (or "intuitive idea") even though it is heavily loaded by frequent use in different connotations by different authors. It is part of our imagination and is mainly to be explained by reference to the complementary role it plays with theory.

Learning a theory necessitates similar cognitive efforts to developing a new theory. Just as intuitions help the researcher to find his or her way to construct new concepts or new relations between existing concepts, there should be some similar potential of imagination that helps to form abstract concepts within the learner's mind. There may or may not be some roundabout ways in which young researchers find access to the common pool of ideas, but teaching should step in to help learners obtain access to specific concepts (and skills), and their cognitive behaviour should thereby improve.

Thus one has to take this complementarity as a challenge. Although one cannot separate these opposed concepts, it might be fruitful to develop a dynamic interplay between them which would help learners gradually to revise their ideas until they finally incorporate some part of those intuitions shared by the community. Fischbein's approach (Fischbein, 1987) seems to be unique in this field. He speaks of raw primary intuitions which are the base for the first mathematical penetration leading to a mathematical model. The raw intuitions are not thereby fully covered, and they are changed to some sort of secondary intuitions which then might form the base for further steps of mathematisation. For teaching, it is of importance that a direct link is established from the primary intuitions to the first mathematical model, and from there to those intuitions which should emerge out of this mathematics. There is an ongoing revision of intuitions and mathematics, a dynamic interplay which should help to understand the abstract theory in the end.

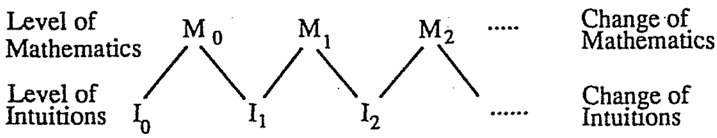

FIGURE 1

The primary intuitions possibly being very vague, it is easier to communicate secondary intuitions via the mathematical concepts to which they are related. Primary intuitions might be a motivating power, or an obstacle to the individual's reconstruction of concepts. The interplay has no interest for mathematics within its structural axiomatic approach, but is vital with regard to learning mathematics.

2. *Intuitions and their role within stochastics*

It has already been stated that one cannot explain specific intuitions without reference to related abstract concepts and vice versa. Nevertheless, some properties of intuitions should be discussed. Intuitions represent immediate knowledge; they are self-evident; they allow extrapolations of conclusions; they are global and synthetic. Furthermore, intuitions bear a strong visual component. They exert a coercive effect on the process of conjecture, explanation, and interpreting. Thus they direct comprehension and action. This network of relations includes belief systems and cognitive structures; it is everything an individual associates with a certain model. A mathematical model is but an objective and communicable representation of intuitive conceptions.

It seems to be important to establish a direct link between intuitions in the learner's mind and these abstract concepts. As intuitions have a long-lasting effect on our cognitive behaviour, teaching might be improved by helping individuals to develop their network of intuitions. This should not only help to motivate them, but also to improve their insight into abstract concepts.

Tracing an individual's understanding of abstract concepts is a precarious task. We have drawn on the dual relations between intuitions and formal theory to clarify matters. There are different approaches to the didactical reconstruction of concepts founded on the complementarity between actions and reflections, beginning with Piaget's "abstraction reflechissante" (Piaget and Inhelder, 1975), and refined later by Dawydow (1977), Doerfler (1984) and others. The framework we have chosen meets many features specific to stochastics. Of course, it is possible to construct links from the former complementarity of intuitions and mathematics. Intuitions accompany actions and also emerge from actions. These intuitions facilitate or hinder the development of a theory referring to the action at the outset.

An axiomatic approach claims not to take any intuition into consideration. The complete separation of the development of theory from intuitions should help overcome the traps of intuitions when they predict false global conclusions with high confidence. The axiomatic body of theory should clarify the puzzling world of intuitions and is therefore thought to be free of any concrete interpretation. Thus the points and lines of geometry could be replaced by beer sets and tables as long as these objects still satisfied the axioms, as a mathematician once joked.

It is remarkable that a revival of the fierce controversy on the foundation of statistics coincided with Kolmogorov's axiomatic foundation of probability. Subjectivists such as de Finetti tried a different system of axioms leading to a different theory. Today the only differences seem to be the basic sets of axioms, the syntax of the theories being largely the same for the rival schools. However, this syntax is still interpreted differently. It is simply not true that the structural approach can be freed of interpretation. The controversy may be described as a rivalry between different primary intuitions, which lead to different secondary intuitions, which then lead to different theories on inferential statistics. Thus, the intuitions are relevant in decoding the theory, and even today this is more controversial than in other parts of mathematics.

A further peculiarity of probability may be seen in the multitude of associated puzzles and paradoxes. This is not true to the same extent for other disciplines. These puzzles may be regarded as manifestations of a conflict between different intuitive ideas, or they may be regarded as points at which intuitive ideas and mathematical theory break away from each other.

3. The tension between order and chaos: a coin tossing example

One can imagine a sequence of several levels of mathematical theories, from one's first mathematisation, with its own set of intuitions, through several stages of refinement, to a final stage of mathematics and related intuitions which are agreed upon by the scientific community. But it would be erroneous to infer that one could proceed smoothly along this sequence, refining at each stage one's set of intuitions in order to progress to the next level of mathematics. There are several intuitions in the individual's mind, and there are many "theories" competing for the individual's choice. Thus, the progress between the intuitive and theoretical level is not straightforward.

This statement remains valid not only for the individual's reconstruction of the concepts, but also, upon closer examination, for the historical development of concepts. Both are illustrated in the figure below, the upper half of which corresponds to the level of theory, the lower half to various intuitive ideas.

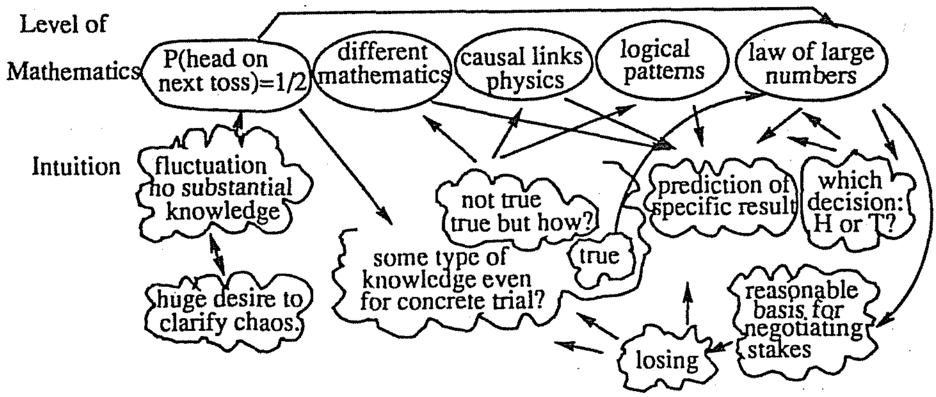

FIGURE 2
Scenario for the interplay between intuitions and mathematics

Let us discuss these ideas within the context of simple coin-tossing. What does it really mean, the theoretical statement, that

P(Head on *next* toss) = 1/2?

What type of knowledge does it constitute for the outcome of the next trial?

The statement seems to be in sharp contrast to the intuitively-felt inability to make specific predictions of this outcome. The individual might already have experienced the highly irregular pattern of "heads" and "tails" in sequences of coin tossing or comparable situations like throwing a die. There seems to be a fluctuation, like a chaos. He or she might already have some experience with (idiosyncratic) strategies for handling the problem and witnessed their weakness. There is "obviously" no substantial knowledge in the situation which might help to improve one's situation.

On the other hand, it has to be stated that there is a big desire on the intuitive side to get this chaos ordered, to overcome this uncertainty. If an individual cannot clarify at the intuitive level what is meant by the statement above that a "head" has a probability of 1/2, or cannot believe this to be an answer to the real "need" of a prediction for the next outcome, then different mathematical "theories", causal links, logical patterns, or even astrological links (or a combination of several of these) will be activated to order that chaos.

For example, the causal links theory might seem promising, as one has an overall impression of its power. These links might even be reinforced in elementary probability by the heuristic motivation of a stochastic *experiment*, being repeatable under the same conditions. Now, what differentiates a stochastic experiment from a physics experiment? Why do probabilists not study the physics of coin tossing and end up with such statements as "If you toss the coin at that angle, with that speed, with ..., then it will turn up heads"? This conflict between intuitive demand and theoretical offer causes a lot of confusion. As an example, if the coin tossing context is slightly changed to counters with red and blue faces, and if it is stated that the red side is held upwards when throwing the counter in the air, then Green (1983) reports that only 75% of the 16-year-olds gave the correct answer. From my own interviews with mathematics students I can confirm that this phenomenon is likely to occur, especially with bright students with a keen interest in science.

Another approach lies in the search for a pattern. This idiosyncratic "theory" is at least promising in that it leads to a specific "solution" to the prediction of the next toss. It is highly interwoven with magic belief and astrology (the law of series, a change is overdue, etc.). These efforts are legion, especially when high wins are at stake, as in the state lotto. This search for patterns in the past series fits a very common and useful approach of behaviouristic learning, e.g. the learning of language by a baby. A specific pattern is also apparent in the events and seasons of the year, etc., and familiarity with such rhythms is vital, especially for children's progress. Thus there is a high incentive to search for such patterns.

Even when one has confidence that "P(H) = 1/2" is establishing some sort of knowledge for a specific toss of a coin, there still remains the problem of applying this knowledge to the next trial. There is an endeavour to reach the law of large numbers in teaching, be it via mathematical derivations or by simulation or just as a heuristic explanation. This should establish secondary intuitions that in the long run the relative frequencies vary around the value of 0.5, and the longer the series the smaller the variation. As every single trial is only a representative of the many similar, interchangeable experiments, one typical case for the general "law", this knowledge might be utilised again for the next toss of the coin. (In some sense again the single case is treated as in science.)

However, this is not a very easy secondary intuition: it is a whole bulk of images which are put together. Shortcuts to it, or less carefully prepared ways to these intuitions, might lead to wrong associations. A confluence of these with the ever existing demand for specific predictions, is the following: In case of HHH HH in the past series, one would then choose T in order to get the relative frequencies nearer to the value of 1/2 of the probability statement, as "the law of large numbers suggests". In fact, Green (1983) reports that 20% of the 16-year-olds chose answer T. Many of my mathematics students chose T, some of them were even willing to accept unfavourable stakes in a bet on it.

A secondary intuition for this probability of 1/2 may result from recalculating it as odds of 50:50. This would allow for equal stakes in a single bet. Thus even if a probability statement has not the ability to predict a specific outcome, it is nevertheless a reasonable basis for negotiating stakes between parties to a bet. (A more applied variation of such a bet is an insurance contract.)

The question of whether to prefer "heads" or "tails" at the next toss is an action-oriented one, and is very close to the intuitive level. The allocation of a probability of 1/2 to a result of "heads" is highly reflection-oriented and it aims at a judgement of how the world might be, and not of what will happen next. Stating the questions within a reflection frame shifts them to the theoretical level, which makes them more difficult to solve. Questions are *loaded* in this respect. Answering behaviour differs according to framing as my own interviews show. However, there is a strong tendency to reframe items in an action-oriented way.

If a reflection on the probabilities yields odds of say 50:50, then there is still the action-oriented question of the strategy to find out what will happen next. If in this case the teacher/interviewer urges the subjects to justify their specific choice, the theoretical demand of such a question and the intuitive understanding (how can a 50:50 chance really give a justification for a choice of say H?) no longer match. Such a conflict is likely to lead to a complete breakdown in communication.

Moreover, unlike other branches of mathematics, the feedback from outside, from real experience, is very weak and indirect. Suppose you accept P(H) = 1/2 as a meaningful statement and develop such an idea as odds for the basis of negotiating the stakes; but then one of the partners in the bet experiences the fact that even with the best understanding of the situation and a good strategy, one can lose. What was it that caused the loss? Was it the fault of one's understanding or the strategy chosen? How does one convince a person of their error who has won in the state lotto with a "nonsensical" strategy?

4. Implications for teaching

Apart from addressing specific misconceptions in class, there is an urgent need to establish direct links between the intuitive and theoretical levels in order to facilitate understanding of abstract concepts. Education in probability should help to structure intuitive ideas, to get this world of vague intuitions ordered. However, without developing this interplay explicitly there is no hope for such a transfer. In order to get the students' intuitions fully involved, I have tried two different strategies.

One of them was to start introductory probability by interviewing students. These interviews were not only intended to disclose their intuitive thinking but had a primary focus on developing the interplay and obviating some of the major obstacles on the intuitive side. There were some passages when I deliberately provoked some of their misconceptions; here I obtained the impression that they already relied on "adequate" intuitions in their answering, but still seemed to be quite uncertain about it.

The second strategy was to develop consciously the secondary intuitions related to the mathematical concepts and to introduce concepts which are intuitively helpful. Only short hints for that can be given here. The idea of weighting different uncertain possibilities by probabilities (search for arguments to allocate equal or unequal weights for these possibilities) should overcome the striving to predict the exact outcome, and its connected causal strategies. The concept of odds facilitates comprehension in the field of conditional probability and Bayes' formula, as it directly challenges related confusing but deep-seated causal ideas. The change in attitudes was obvious and facilitated teaching the more refined concepts later on.

Empirical evidence indicates that there are different types of approach which are differently accessible at the intuitive level. Logical thinking is hard but once one has recognised the underlying logical relations, then it is fully accepted; causal thinking is intuitively very convincing; stochastic thinking, however, is very theoretical. Teaching has to develop secondary intuitions that clarify how this stochastic thinking relates to the other two approaches.

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