REVISING PROBABILITIES ACCORDING TO NEW INFORMATION: A FUNDAMENTAL STOCHASTIC INTUITION

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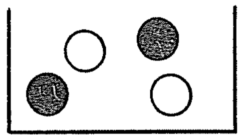
Probability judgements may have to be revised if new information is available. From the mathematical perspective probability revisions are intimately connected to the notion of conditional probability and Bayes' formula, a subsidiary concept and a trivial theorem respectively. Nevertheless empirical investigations in subjects' understanding of probability do indicate that people do not cope adequately with situations involving probability revisions, if they have been taught the mathematical concepts or not does not matter.

In what follows I will try to sketch some phenomena of misunderstanding, give some comments on the interplay between mathematics and intuitions which I think represents the origin of lack of comprehension. A brief overview on the favor concept should enable the impression that by way of teaching this concept probabilistic reasoning could be improved.

1. Examples - Phenomena

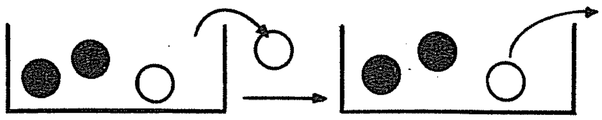
Example 1: "Falk Phenomenon"



Two marbles are drawn without replacement from the urn above. The question for the probability to get a white marble at second try if one has drawn a white one at first try, let us denote that by $P(W_{II}|W_I)$, does not cause any particular problem and is usually answered with 1/3. However, many subjects fail to solve the "reverse" problem, namely to evaluate $P(W_I|W_{II})$ (see Falk, 1983). They will argue: The colour of the second marble cannot influence that of the marble drawn at first. This *missing causal influence* induces them to think as if W_I be *stochastically independent* of W_{II}.

There is an intermixture of probabilistic reasoning and causal inference. Two of many possible reasons for that are:

a) Conditional probabilities are usually applied in "forward looking" situations, at least until Bayes' formula is dealt with. An urn represents the situation and the probability to calculate:



$P(W_{II} | W_I) = \frac{1}{3}$!

Causal influence is associated in two senses: The first try changes the urn – the urn "causes" the new probability.

b) Motivating and defining the mathematical concept of independence is difficult. The following plausibility argument should help: "If you have a situation with missing causal influence then you can model this by stochastical independence." The example above shows that things are much more complicated.

The favour concept is intended to help discriminating probabilistic reasoning from causal thinking.

Example 2: Simpson paradox

In 1973 admission rate of female applicants (35 %) was smaller than that of their male colleagues (44 %) at Berkeley University. Searching for the reason of this "discrimination" of sex it turned out that in some departments women had higher admission rates than men, most of the departments had similar admission rates. But: Is it possible that admission rates for *all* departments are higher for women than for men, and nevertheless being lower for women for the whole university? The following example is from Falk and Bar-Hillel (1980):

	Ad		Ap	
	Female	Male	Female	Male
P	0,8	0,4	0,1	0,9
M	0,12	0,6	0,9	0,1
Overall	0,19	0,37		

P Painting

M Music

Ad Admission rate

Ap Application rate

Many people are baffled by this phenomenon. One reason could be that they transfer relations which hold for logical reasoning but not for probabilistic thinking: Under the restricting hypothesis P information female increases admission rate Ad, likewise for M. Thus the following relations hold:

(1) P Female $\Rightarrow$ Ad $\uparrow$

(2) M Female $\Rightarrow$ Ad $\uparrow$

} "Now it is easily derived that"

(1') P Female $\Rightarrow$ Ad $\uparrow$

(2') M Female $\Rightarrow$ Ad $\uparrow$

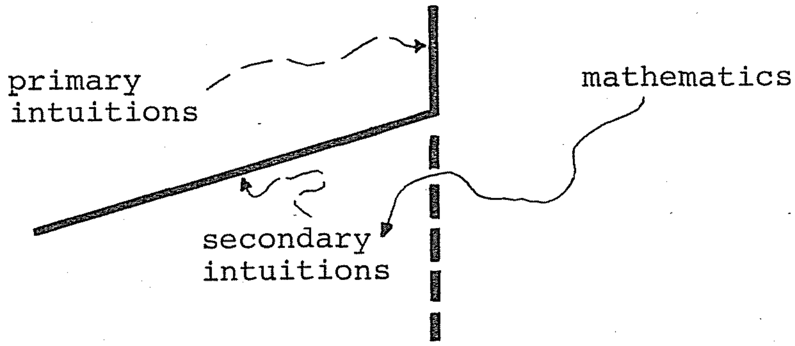
$\Rightarrow$ (P $\wedge$ Female) $\vee$ (M $\wedge$ Female) $\Rightarrow$ Ad $\uparrow$

The conclusion is perfectly right if one deals with logical implications but for conditional probabilities (1) and (2) are not equivalent to (1') and (2'). All the other steps of derivation would hold for conditional probabilities, too.

Probabilistic reasoning is by no means a weakened form of logical inference. It has a different formal structure. An investigation of the favor concept should make this obvious.

2. The interplay between mathematics and intuitions

Relations constitute notions, concepts, and methods. *One* quality of relations are those which connect notions within a mathematical theory. But the spectrum of relevant relations is much more varied: How to apply a certain model in a real situation? How to justify a special type of theory in favor of concurring ones? Which intuitive imaginations guide/hinder the development of a theory as primary intuitions, be it the objective development as it is communicated e.g. by textbooks, be it the subjective reconstruction of such a theory within the subjects' cognitive structure. Which secondary intuitions might/should emerge out of a theoretical treatment of a problem field which could be guidelines for easy understanding of the theory and for direct and meaningful application of it in other situations?



I hold the view that there is a lack of tight connections between primary intuitions and mathematics and between mathematics and secondary intuitions. Stochastics is learned and taught as formal, meaningless body of symbols (the mathematical theory itself or recipes for application) neglecting the full and varied interplay between intuitions and mathematics. The result is a biased or ineffective understanding which is the reason for the fact so often observed that many of the paradoxes remain paradoxical for most of the people even after they have been educated carefully in theory. Relations usually taught do not really allow grasping *what* makes situations paradoxical.

This in mind I give a high priority to efforts in teaching stochastics to develop a *relational understanding*, by which I mean to construct new, other, and wider relations which do allow a *direct* understanding of the formalism of a theory. The favor concept is one such concept which I "favor" to overcome some special lacks of subjects' stochastical understanding. By this concept I want to reconstruct strong and effective secondary intuitions and make clear what makes stochastical reasoning different from other types of reasoning.

3. The favor concept

Definition: Let E and F be events with $0<P(E), P(F)<1$

a) $E \nearrow F$, in words E *favors* F : iff $P(F|E) > P(F)$.

E favors F if the probability of F has to be raised on basis of the information E.

b) $E \searrow F$, E *disfavors* F : iff $P(F|E) < P(F)$.

c) $E \perp F$, F *independent* of E : iff $P(F|E) = P(F)$.

No probability revision necessary in this case.

Theorem 1: $E \searrow F$ iff $\bar{E} \nearrow F$.

Reversing the information induces reversion of the direction of necessary probability revision.

Theorem 2: Either $E \nearrow F$ or $E \searrow F$ or $E \perp F$.

There is a *trichotomy* of possibilities: Either an information E induces a probability revision upwards, downwards or it is of no influence for the probability judgement of F.

Theorem 3: $E \nearrow F$ iff $F \nearrow E$. (likewise for $\searrow$, $\perp$).

This *symmetry* confronts the favor concept with deep seated inadequate primary intuitions. At first glance many subjects are prompted to look for a counterexample against the symmetry of $\nearrow$. Understanding this symmetry is in my opinion one of the central issues in probabilistic thinking.

A formal but easy proof makes clear that information E and event F play a symmetric role from point of view of probability judgements. Moreover this symmetry is in line with following intuitions: If you think of E_1, E_2, . . . and E_r as possible "explanations" for event F then some E_i will favor F, some E_j disfavor F. If you observe F then it is straightforward to think of just those explanations E_i which favor F as being now more plausible (probable).

By means of this symmetry the inadequate causal strategy in the Falk phenomenon is easily detected, W_I cannot be stochastically independent of W_{II}, as W_I disfavors W_{II}!

Theorem 4: e.g.: $E \nearrow F \wedge F \nearrow G \not\Rightarrow E \nearrow G$.

Nontransitivity is one of the relations of favoring which discriminates it from logical inference but also from causal schemes. One cannot replace F by an information E which favors F.

For more details on favoring which also explain the Simpson paradox see Falk and Bar-Hillel (1983) and Borovcnik (1985).

4. Concluding remarks

I could give a sketch of the favor concept. My intentions to include this concept in elementary probability are manyfold: To get a better, more direct, intuitive orientation in situations involving probability revisions and even a deeper comprehension of Bayes' formula (see remarks to theorem 3). By analyzing the formal structure of favoring one could make clear the peculiarities of probabilistic reasoning. To contribute to the interplay between intuitions and mathematics, thereby integrating stable stochastical strategies in the learning individual.

The relations of favoring are nonstandard but could enable what I call *relational* understanding.

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