RANDOM WALKS AS LEARNING SPROUTS IN THE DIDACTICS OF PROBABILITY

Jorge Soto-Andrade[1] and Daniela Díaz-Rojas[1]

CIAE and Department of Mathematics, University of Chile

We argue that random walks are a foremost avatar of the key notion of randomness and that enactive metaphoric "learning sprouts" may be implemented to facilitate their learning by non mathematically oriented students, from elementary school onwards. We discuss the underlying theoretical background and present concrete examples involving a broad spectrum of learners.

INTRODUCTION

Randomness is a key notion in mathematics as well as mathematics education. In fact probability and statistics are making their way nowadays into elementary school levels in many curricula in the world. Nevertheless the way they are traditionally taught rarely arouses interest or probabilistic intuition among the millions of children in the world who are inescapably exposed to them. Our theoretical framework for mathematics education, based on metaphoring and enaction, didactical situations and learning sprouts, suggests instead an *enactive metaphoric approach* to randomness, that we exemplify in what follows.

Random walks are undoubtedly the foremost avatar of randomness. Indeed, they cross boundaries, arising in the "natural world", e. g. Brownian motion (Powles, 1978), mosquito flights (Pearson, 1905), foraging patterns in human hunter–gatherers (Raichlen et al., 2014) as well as in the "cultural world", e. g. construction of random hexagrams when consulting Yi Jing (Wilhem, 1956, Marshall, 2015) or Saint Francis friars random pilgrimages in medieval Italy (Soto-Andrade, 2013; Anonymous, 1600). They are a visual embodiment of randomness, easily enacted and simulated, starting at primary school and generating "learning sprouts" that unfold up to graduate teaching. We can approach them in manifold ways: statistically, metaphorically, probabilistically… They provide "universal models" and metaphors for sundry phenomena. Indeed, simple random walks may be explored "bare handed", with no previous statistic or probabilistic tools or concepts. Students may tackle the paradigmatic impossible question: *Where is the walker, after a given number of steps?* with sheer common sense. Most important, they realize quickly that there are several levels of answers to this sort of impossible question, from: "nobody knows!" to quantification of the degree of likelihood of the different possible places where the walker could be.

After shortly recalling our theoretical background, we describe briefly below some concrete examples of "learning sprouts" for random walks implemented with a broad spectrum of learners, that are novel in the literature and have not made their way yet to the curriculum.

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THEORETICAL BACKGROUND.

Metaphors are not just rhetorical devices but powerful cognitive tools, grounded in our perceptive-motor system that help us understand or build new mathematical concepts ((English, 1997; Lakoff & Núñez, 2000; Manin, 2007; Sfard, 2009; Soto-Andrade, 2006, 2014) and solve problems in an efficient and friendly manner (Libedinsky & Soto-Andrade, 2015), fostering the democratization of the learning of mathematics. For instance the concept of probability emerges when we see a symmetric walker on the integers *splitting* into two equal halves instead of going equally likely right or left (Soto-Andrade, 2013). Machado's verses: "Wanderer, your footsteps are the path, nothing else; wanderer, there is no path, you lay down a path in walking" are a metaphor for enaction (Malkemus, 2012), as pointed out by Varela when introducing enaction in cognitive science (Varela, 1987, p. 63; Varela et al., 1991). Bruner (1953) introduced enaction in math education as "learning by doing" and characterized *enactive, iconic and symbolic representation modes*. For recent developments see Masciotra, Roth & Morel, (2007) and Proulx & Simmt (2013). Our main hypothesis (tested in the activities below) is that most students can think mathematically if they *enac*t first suitable didactical situations, involving problem posing and solving.

The theory of didactical situations (Brousseau, 1998) is the unfolding of the *emergence metaphor:* mathematical concepts or procedures to be taught should emerge in a suitable challenging situation the learner is enmeshed in, as the only means to "save his life". No real learning is possible when mathematical concepts "come out of the blue" or are "airborne". Key metaphors are likely to emerge as sparking voltaic arcs among learners when enough "didactical tension" builds up in a challenging didactical situation.

Fundamental mathematical ideas, like randomness, run across all school grades, unfolding through enactive didactical trajectories, called "learning sprouts", closely related to the trajectories of Wittmann (2012), that are however usually seen as linear sequences of steps to be followed in a predetermined order ("staircase metaphor", NAEYC, 2002). Our viewpoint is closer to Dall'Acqua's (2010), who says that the curriculum is a fluid, enactive and unpredictable process emerging from the situated relationship between students and teachers.

EXPERIMENTAL BACKGROUND (THE LEARNERS)

a) In service primary school teachers enrolled in a 15 month math professional development program, at the University of Chile (2007–2015, 15 to 30 teachers per year, and their students).

b) 1st year University of Chile students majoring in social sciences and humanities, from 2007 to 2012 (1 semester mathematics course, 50 students per course, on the average).

c) University of Chile undergraduates majoring in mathematics and physics (one semester probability and statistics course, averaging 30 students) in 2009-2011 and 25 undergraduate and graduate math students in a common course in 2015.

d) University of Chile prospective secondary school teachers (one semester probability and statistics course, 50 students on the average) in 2009 -2015.

e) Elementary and secondary math school teachers participating in various one week professional development courses, in Chile (2006-2008, 2011-2015, 30 teachers per course)

f) Juvenile offenders in a social re-insertion program (Univ. of Chile, 2012-2014, 7-8 per cohort)

METHODOLOGY.

Learners were observed by the authors during work sessions, their written outputs were kept or scanned. They also answered questionnaires related to their use, appreciation and preferences regarding metaphors used in courses a) and b). Learners a) and e) did group work most of the time. Other learners participated in interactive lessons with high horizontal interaction.

EXAMPLES.

1D Baby Brownian motion for elementary school: a frog's symmetric random walk on a row of 9 stones in a pond, up to 4 jumps, in a 5th grade class, 30 students (López, 2014). The teacher belongs to group a) above; her students enthusiastically explored and enacted the walk in the school basketball court and realized quickly that not all stones were equally likely. Working in groups they figured out the corresponding probabilities as "3 out of 8" and so on.

2D Baby Brownian motion, with learners a) to e). Secondary math teachers and elementary general school teachers are more prone to expect all possible corners to be equally likely than prospective teachers. See Soto-Andrade (2013) for more details and experimentation up to 2013.

Expected waiting time for success in a dichotomic random experiment: Looked upon as expected stopping time for a random walk with one absorbing barrier. Students metaphorize the problem as waiting for heads when tossing coins and enact it in circles of 20 to *see* that the ideal average waiting time should be 2 tosses (staring at all the coins on the floor). For learners b), d).

False positives in biomedical tests: Awesome Bayes theorem becomes dispensable if you metaphorize this problem as a two step random walk. If we then employ a pedestrian metaphor (Soto-Andrade, 2013) for the random walk, we come to the natural frequencies approach suggested by Gigerenzer (2011). Tested successfully with learners b), c), d).

The ruin problem: Proposed to learners c) in 2015. Working in groups undergraduates and graduates alike metaphorized the problem (first for fortunes 2 and 3) as a random walk on a grid, took advantage of an hydraulic metaphor and solved in several ways (calculating a normalized Fibonacci series as well as avoiding series calculation by a clever fractality argument leading to a harmonic function view point) Metaphoring enabled undergraduates to keep the pace of graduates.

DISCUSSION

In the course of our didactical experimentation, we have seen that enacting a problem or situation facilitates the emergence of new ideas and insights in the learners, that otherwise would be out of reach for them. Also they develop the skill of looking at a problem and seeing something else, especially that realizing that a wide spectrum of problems may be seen (metaphorized or modeled) as random walks, and that this enables them to solve them in an intuitive and friendly way. In several cases (expected waiting time, ruin problem) an enactive approach enables the learners to find alternative ways that avoid computing unfriendly infinite series (whose value they gleaned a posteriori!). In retrospect we made progress in implementing learning sprouts for random walk all the way from elementary school to advanced graduate math courses.

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