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UNDERSTANDING CHILDREN'S CONCEPTION OF RAMDOMNESS THROUGH EXPLORATIONS WITH SYMMETRICAL POLYHEDRONS

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In this study an attempt has been made to understand middle grade Indian school students' (aged 13-14 years) perceptions related to generators of randomness, while they engaged in an open-ended exploratory setting. Detailed discussions of two groups of students who worked in small group situations, to choose, from a set of eight symmetrical (in perfect or as partial) polyhedrons, a shape that according to them would be optimal for playing the games of chance, have been provided to bring to the fore different meanings that students ascribe to for qualifying an object as a random generator. The results indicate that students rely on physical properties of the shape and size of sample spaces for making judgements. They also held misconception related to exhaustiveness of sample space.

BACKGROUND

Randomness as a concept is challenging to define. Studies that have contributed in understanding the notion of randomness in children (Piaget & Inhelder, 1975; Green, 1983; Batanero & Serrano, 1999; Metz, 1998; Jones, Langrall, Thornton & Mogill, 1999; Watson & Moritz, 2003; Pratt, 1998; Langrall & Mooney, 2010) have inextricably focused on uncertainty or unpredictability and having no-pattern as indicators for qualifying an object for a generator of randomness. In all these and similar studies, children were provided with a cubical dice and their responses while experimenting with it were analysed to know their thoughts, conceptions and misconceptions. Studies have also brought to the fore the irrational beliefs that children hold regarding outcomes of a standard cubical dice. They consider luck (Amir & Williams, 1999), technique of throwing the dice (Green, 1983; Pratt, 2000), liking towards some particular numbers (Vidakovic, Berenson & Brandsma, 1998; Watson & Moritz, 2003), difficulty in getting a six (Green, 1983), God, fate, or mental powers (Truran, 1995) as being responsible for biased outcomes. Fischbein (1975) defines such cognitive attributes that are spontaneous, global and self-evident to the believer as primary intuitions. He claims that since these beliefs are mostly derived from everyday experiences, they are difficult to challenge and correct. He further asserts that through systematic didactic situations such intuitions can be restructured. Through appropriate pedagogic experiences primary intuitions can be challenged, modified and linked to formal mathematics.

In an attempt to interpret and implement Fischbein's ideas into more practical forms, two perspectives of research in probability have emerged. One focuses on identifying primary intuitions, heuristics, biases and misconceptions that people hold while doing probability (Kahneman & Tversky, 1972; Konold, 1989), while the other takes a constructive – interpretative path for understanding the reasons behind people's actions when they encounter stochastic situations (Nilsson 2004, 2007). Proponents of the latter view believe that since knowledge is embedded in contexts, actions performed in a situation reflect the cognition. Actions, and thus learning is grounded in the situation in which it is acted. How children react is an interplay of their cognition and the context in which learning occurs (Cobb, Yackel & Wood 1999). Linking this idea to the learning of probability, one can argue that children's actions under specifically designed probability environments act as gateways for knowing and interpreting their thoughts, intuitions and possible reasons for their (mis)conceptions. Opportunities of self-explorations stimulate children's informal intuitions. So questioning and interpreting children's arguments while they make judgements might help educators map children's naïve conceptions to the formally established theories of probability.

We also know, from historical anecdotes, that cubical dice were not the only tool used in the past for making decisions in occasions of chance. Human beings have been dependent on an assortment of devices such as heel bones of hoofed animals (astragalus), especially carved stones, specially leafed rosettes, spheres with twelve-fourteen knobs, shells, tetrahedral dices, rectangular prisms, triangular sticks, coins and hexagrams for not only playing games but for also making vital decisions and soliciting divine direction. Most of these objects exhibited perfect symmetries. Certainly people had perceived a relationship between symmetry and randomness (Bennett, 1998; Sautoy, 2008). If symmetry is one of the considerations for choosing a device for anticipating events related to chance, it is justifiable then to ask if any other shape, apart from the cube, that inherits properties of symmetry, can also be used for generating events of chance.

In line of the above, that is, children's incomplete understandings related to randomness, their biased opinions against a fair cubical dice for a perfect object of randomness, a need to embed contextual experiences to know children's notions and the evidence provided by historical anecdotes attesting use of symmetrical objects as dices, in this study unconventional resources, in an explorative setting were provided to know how children make meaning of randomness through symmetrical objects. In particular, it is hypothesized that providing an exploratory, open-ended approach that gives a freedom of expression, will be instrumental in understanding children's reasoning, judgement and sources of (mis)conceptions related to randomness and random generators.

THE STUDY

This paper presents results of a microstudy that formed part of a larger project on probability in India. This episodic phase was based on the premise that giving children opportunities to create randomness, rather than perceive it, will help in understanding their beliefs related to the concept. A dynamic process was adopted where students were given open opportunities to explore and experiment with various symmetrical polyhedrons and choose one for playing games of chance. It was postulated that in the process of choosing their dice, the students will inadvertently bring in their meanings of randomness.

A student-oriented approach for understanding students' notions of chance encounters was adopted. An attempt was made to know what students would actually do in a probabilistic situation which is governed by the objects of randomness. The endeavour has been to interpret the different meanings that students ascribe to in order to qualify an object for a generator of randomness.

METHOD

Participants

This study took place with eighth grade Indian students (age group 13-14 years) from a Delhi (India) private school. In all, there were 8 students divided in two groups, with four students in each group. The results follow a case study approach to draw out, from the arguments and discussions of the groups, the commonalities in children's internal thoughts related to dices as objects of creating randomness.

In the Indian curriculum of Mathematics (NCERT, 2005) the concept of chance is introduced in grade VII, so all the students of the current study were familiar with elementary concepts related to probability. Most of the content of probability in Indian textbooks follow the classical interpretation of finding probability (Gandhi, 2015) so the students knew theoretical ways of finding probability of simple events. The students had also, as part of their regular teaching course in probability, experienced cubical dice and coins as objects of randomization.

Data Sources and Experimental Setting

The major source of data included the eight polyhedrons that show properties of symmetry in perfect or as partial. The children had already experienced these polyhedrons as part of their geometry curriculum of classes VII and VIII (NCERT, 2005). The eight polyhedrons included four platonic solids that show perfect symmetries: regular tetrahedron, cube, regular octahedron, regular icosahedron; two pyramids: one with a square base and the other with a regular hexagonal base; and two prisms with the top and bottom bases being regular triangles and regular hexagons respectively. While making the non-platonic polyhedrons care was taken to ensure maximum regularity in the lateral and base faces.

In the experimental setting both the groups were made to sit apart from each other and work independently. A set of eight polyhedrons was provided to each group. The participants were encouraged to work within their small groups and decide which from the given polyhedrons would they use for paying games of chance, such as ludo (a common board game played by Indian children)? After they had analyzed and experimented with their chosen object they were asked to state the reasons for their choice. The aim was to note how students rationalized their decisions regarding their choice of dice and the inhibitions that they carry (if any) against the standard dice, i.e. cube.

Children's workings were video-recorded and later converted into transcripts. My role was of an active observer, intervening as minimum as possible to only seek clarifications. As the setting was designed to be an open-explorative one, the students were encouraged to take as much time as they wished to. However, it was noted that both the groups took near to 70 minutes for selecting and making decisions.

A qualitative analysis was done on students' responses and actions to identify their emerging ideas that reflected their understandings. Each group brought different aspects to ascertain randomness of their selected dice. In this study I'll focus on the common threads between both the groups.

RESULTS AND DISCUSSIONS

In this section students' actions, discussions and interpretations of how they made sense of randomness from various polyhedrons given to them will be presented. Students worked in their respective groups to select a shape that they found most suitable as a dice.

It was observed that before starting to base their arguments, each group spent enough time in investigating physical properties of all the shapes. During this initial inspection they examined each shape by throwing, spinning and rolling. Though none of the groups made any specific comment, it was evident from their explorations that their focus was on judging the orientation of the shapes. These activities revealed that the students were trying to explore the free motion of the shapes. In fact, the participants iterated on this examination several times during the intervention phase.

After the initial investigation, both the groups were ready with their respective dices. One of the groups (further referred to as Octahedron Group or OG) chose the regular octahedron and decided to mark its faces from 1 to 8 while the other group (will now be referred to as Icosahedron Group or IG) selected a regular icosahedron and marked its faces with numerals 1 to 20.

It appeared that for both the groups a paramount condition for selection was associated with the size of the sample space that their dices would produce. Both the groups ensured that their respective dices had larger sample space (as compared to the cube). Larger sample space meant smaller probability of each outcome ($\frac{1}{8}$ and $\frac{1}{20}$ respectively) which meant wider possibilities of getting different numbers (8 and 20 respectively). Both the groups ascertained that since their respective objects had more faces (as compared to the cube), there will be larger sample spaces resulting in smaller chances of predicting the outcome. The pupils had connected the size of sample space with the idea of unpredictability. Larger sample space corresponded to lower chances of knowing the outcome yet to occur. Increased sample size gave them the confidence that in their respective dices the probability of each outcome being less, though derived theoretically, would result in larger uncertainty of knowing the number yet to occur. An example of their discussions reflecting this understanding follows:

IG: This has 20 faces. We have more numbers to put on each face so nobody will be able to know which number they will get in the next throw.

OG: There are 8 faces. The probability of each face is $\frac{1}{8}$, which is very low.

Next, for both the groups the perfectly symmetrical shape of the object established fairness. Though no exclusive discussion took place nor was there any need for clarification, but students' choice of picking only perfectly symmetrical shapes, having congruent faces had, covertly, indicated their notions of fairness. Selection of only regular polyhedrons was an evidence substantiating children's intuition of associating physical properties of shapes with the idea of fairness. For them, since the faces of their respective dices were same, all faces will 'show up' once in finite throws. Both the groups had approached fairness through visual appearance of perfectness of the shapes. The physical symmetry of the objects was also a powerful tool in ensuring free fall of the dice after the act of throwing. The groups attested that since their objects had congruent faces, the dices would land freely without any disruption from any external forces. Following discussions of both the groups exemplify these results:

IG (Student 1): There are so many faces that it will land on a different face each time. All faces are equilateral triangles…it can land on any of these.

IG (Student 4): We can throw it many times. Every face is same so each time it (number that showed on the top most face) is different.

OG (Student 3): In this diamond shape (octahedron) it is very difficult to do so. There are so many faces. It is so difficult to say which one will come up. Any of them can come up anytime.

OG (Student 2): The shape can land anywhere. See...[throws the dice up in the air and allows it to land freely on the working table. During landing the piece was unstable for a short while but immediately turned slightly before resting finally. This instability in landing resulted in different numbers while and after landing].

What also appeared as a common thread across both the groups was a tendency towards equiprobability bias. Both the groups exhibited a bias towards getting every number at least once during the game, in finite throws. They intuited that irrespective of the order of occurrence of numbers, it was certain that each face will occur once eventually. In addition to the equiprobability bias the students also held, what I call, a misconception related to 'exhaustiveness of the sample space'. The students believed that each number should occur at least once before an already obtained number reoccurs. They held that all the numbers of the dice have to occur once before any of them reappears. This exhibits that they intuit probability of happening or not happening of events as being cyclical. Irrespective of the order in which the outcomes appear, it is certain that each outcome will occur at least once. An example of children's bias towards equiprobability and exhaustiveness of sample space could be traced through discussions that took place in the Icosahedron Group. The IG group listed all outcomes and accounted for all faces to come up at least once:

IG (Collective): If we use this dice the game will end fast

Researcher: What do you mean by 'fast'?

IG (Student 2): In all there are 20 numbers, each of them will occur once in twenty throws. So, if four people are playing, the game will end very early. Once in twenty throws they will get a 20..., or a 19, ..., or a 15 which will lead to bigger jumps.

IG (Student 4): There are 20 numbers so one of the player will get a 20 within twenty throws. That player will go much ahead…. And if he gets another 20, he will be close to winning the game. Game would end early i.e. for one in 20 times one would get a 20. So the game would end in 20 throws.

It was also evident that students perceived the idea of probability being connected with proportionality. Since these children had mostly been taught through the classical approach of probability, they assumed probability as an extension of proportionality, wherein to find the probability of an event one should first enumerate the sample space and then find the ratio between desired the event and size of the sample space. Both the groups claimed that the probability of getting any face in their dices was equal ($\frac{1}{8}$ and $\frac{1}{20}$ respectively), a notion which is in line with the classical definition of probability. Since, through classical approach they had learnt that the probability of occurrence of each event is equiprobable, they calculated probability of each face in a deterministic way.

It should be noted here that, in the set of polyhedrons, though cube was also given to the students, none of the groups selected it as their first choice. Probably, since a cube has fewer faces as compared to other polyhedrons available, the children were more attracted towards these shapes. As a concluding exercise, the students were asked to compare their observations with the cubical polyhedron and comment on its properties that make it a popular object for playing games. Even after their empirical experience with other symmetrical objects, the participant tenaciously clung to their beliefs that a cube will not give equal and fair results and that skilled hands can easily maneuver it to get desired outcomes. It is possible that their prior experiences of playing games with only the cubical dice may have created such prejudices (Watson & Moritz, 2003). Perhaps, children are conditioned to using a cube.

CONCLUSION

In this paper I have aimed to investigate students' reasoning for selecting objects as generators of randomness. Analyses of the two sample groups reveal significant ramifications about children's understandings of shapes and randomness. While substantiating their choices, students not only experimented with randomness but also verified each of the characteristics that qualified a device for being a generator of randomisation. My aim was to build a constructive, explorative domain in which it was possible to work with these formalisations, rather than approach the formal as a separate domain grafted onto activity. My intention, then, was to put individual learners in situations where they could express their thoughts freely without having been conditioned.

From the working of both the sample groups I could draw three major conclusions: First, while basing their decisions children ascribe to the physical properties of the shapes. For children of this age group the structural characteristics of the shape played a crucial role. Aspects related to perfectness of the shapes, physical symmetry of the shapes, number of faces, physical orientations, ways of using the dice - throwing, spinning, rolling, and ensuring lesser possibilities of maneuvering were key components in selecting a dice.

Secondly, the work identified children's perceptions related to size of sample spaces of random generators. Studies done till now have emphasized on strengthening the idea of sample space (Horvath & Lehrer, 1998; Benson & Jones, 1999; Jones et. al., 1999) but none has tried to understand how children make sense of sizes of sample spaces. Through this study I found that for children of this age group the size of sample spaces appeared to be a major criterion for deciding the randomness of the objects. Children considered the size of sample space of each polyhedron and the ones with larger sample spaces were selected. Larger sample space meant lesser probability of each outcome, which, for these students, ensured fairness. In fact, these children connected many aspects related to randomness, such as of fairness and unpredictability, with the size of the sample spaces. Larger sample space ensured better randomness.

The third result draws on the nature of misconceptions and biases that children hold towards dices. Equiprobability bias was again evident in the students' working. The intuition related to equal chances of occurrence of all faces in a fixed number of throws was dominant. Through this study, along with the equiprobability bias, I could also identify a misconception that students hold. It was found that children believe in 'exhaustiveness of the sample space'. They intuited that though each outcome of their dice has to occur once, it will not appear unless all the outcomes have appeared at least once. The outcomes follow a principle of exhaustiveness. A number shows up again only after each one of them has been exhausted. Every outcome will occur once, within a fixed number of throws, before any of them reappeared. Outcomes repeat only after every number has shown up in a cycle of throws. The children agreed that though the order of occurrence did not matter, the occurrence of every number at least once is an essential criterion.

This study also gave an opportunity to trace reasons behind students' views related to the cubical dice. The equiprobability bias along with the 'sample space exhaustive' misconception helped, to some extent, in understanding the reasons behind children's inhibitions against the standard dice.

Children hold the misconception of sample space exhaustiveness even with the standard dice. While playing games if they find that their opponent has already got a 6, they hold their own chances of getting a 6 again in near future as fewer. They believe that in a cubical dice since a number would appear only once in a cycle of six throws, their opponent getting a six recently eliminates their chances of getting it again. They would have to wait for another round to start.

Through the open-constructive spaces that were provided to the students in this study, I could capture how students connected physical symmetry of objects with property of fairness, misconception that they hold related to exhaustiveness of the sample space and emphasis that they give to the size of sample spaces for ascertaining unpredictability and fairness of random generators. Freedom to express thoughts and to explore and experiment with unconventional tools (with which they had not been conditioned to yet) facilitated in tracing students' perceptions and misconceptions related to random generators.

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