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TEACHING 9 AND 10 YEAR OLD CHILDREN ABOUT RANDOMNESS

Peter Bryant, Terezinha Nunes, Deborah Evans, Laura Gottardis & Maria-Emmanouela Terlektsi

Department of Education, University of Oxford, UK

Research on children's knowledge of randomness has concentrated on (1) the link between randomness and uncertainty, and (2) the independence of successive outcomes in random sequences. In our study, children, aged 9-10 years, were taught about these two aspects of randomness in a 5-session intervention which concentrated on the children (a) making predictions about random and non-random sequences (e.g. about throwing fair dice and loaded dice) in familiar contexts (e.g. looking for particular cards in packs which were either shuffled or not) and (b) discussing their predictions and solutions with the other children in the group. These children's scores in an evaluation of their understanding of randomness improved from pre-intervention to post-intervention tests more than those of other children who had not been taught about randomness. We conclude that it is possible to teach 9 & 10 year old children about randomness.

BACKGROUND

Randomness lies at the heart of uncertainty and probability, since all probability problems deal with temporal sequences or spatial arrangements that are completely or partially random. It is a complex concept (Batanero & Serrano, 1999) which children and many adults find difficult to understand, and yet there are some fundamental aspects of randomness, such as (1) the connection between randomisation and unpredictability and (2) the independence of successive outcomes in a random sequence, that even primary school children might grasp since these are part of their everyday experiences. Both these aspects of randomness play a central and essential part in any game that involves dice or shuffling cards, and children often play such games.

Yet, on the whole, existing research points to children's difficulties rather than to their successes in understanding both aspects of randomness. In a classic study of children's understanding of the link between randomisation and unpredictability, Piaget and Inhelder (1975) asked children, whose ages ranged from 4 to 12 years, about the consequences of tilting a tray which at one end held red and white marbles, grouped by these colours. Most of the younger children failed to understand that the separate grouping by colour would most probably break down as the marbles rolled to the other end and that each successive tilt would be likely lead to new and unpredictable spatial arrangements of the two colours. These investigators concluded that, at first, children do not understand how randomising leads to unpredictability.

Research has also consistently documented striking failures in children as old as 10 years in recognizing the independence of successive events in a random sequence after a run of one particular outcome, in a context where the different outcomes are demonstrably equiprobable (Chiesi & Primi, 2009; Fischbein & Gazit, 1984).

If children have such difficulty in understanding these familiar aspects of their own lives, perhaps they could be taught about them, but studies of teaching and learning randomness are hard to find. Some of the children in Fischbein and Gazit's study had previously taken a course on probability, but it isn't clear how much of the course dealt directly with the nature of randomness.

We report here an intervention study in which we taught 9 and 10 year old pupils about randomness with some success. The key features of the intervention were that pairs of children co-operated in solving problems and making predictions about random and non-random sequences in familiar contexts and with familiar objects, and then discussed their solutions and the reasons for their predictions with other pupils and with the adult delivering the intervention.

DESIGN

Seventy-five children from three Primary schools took part. They were 9-years (63.2%) or 10-years (36.8%) old when the project began. The three schools in the project catered for children from quite a wide range of SES. Within each school, the children were randomly assigned to three groups.

1. A group (N=24) taught about randomness (Randomness Group) in 5 weekly sessions
2. A control group (N=22) taught about directed numbers and not about randomness or any other aspect of probability (Directed Numbers Group) in 5 weekly sessions
3. A control group (N=29) not involved in any intervention (Unseen Control Group).

All the children were given a pre-test just before the interventions started, an immediate post-test just after the interventions ended, and a delayed post-test later on. The mean interval between the pre-test and the immediate post-test was 7 weeks 5 days. The mean interval between the immediate and the delayed post-tests was 10 weeks 4 days. The tests were administered in the children's schools.

Intervention

In the first randomness intervention session the activities centred on the effects of randomisation and on the difference between random and non-random sequences. The pairs of children were given, at different times, a shuffled, and therefore randomised, pack of Happy Families cards (a game that is widely played in the U.K.) and an unshuffled pack in which the cards were grouped into families; the order of each member of the family (father, mother etc.) was the same for each family in the latter pack. The children were asked to make predictions about which card was at the top and what would be the next card, and the next after that and so on. Then they played a round of Happy Families with both kinds of pack. They were asked to discuss and to come to a conclusion about the ease of predicting the next card to be drawn from the top of the pack in the shuffled and unshuffled packs. In the second part of this session, the pairs of children were given two counters and then one child secretly took either 1 or 2 counters in their hands. The other one's task was to predict whether the number of counters that held by the first was an odd or an even number. They did this several times. The task is apparently a randomised one, but often one child adopts a consistent strategy (e.g. alternating 1 and 2), which the other child often discovers. At the end of the series of predictions, the children discussed whether either had detected this sort of departure from randomness in their game. The session ended with a discussion among the group as a whole about the fairness and also the degree of randomness of different ways of starting and playing games.

The main purpose of the second session was to extend the children's understanding of randomised and unpredictable sequences. They played some purpose-designed computer games in which they attempt to guess the order in which three pictures will appear. In some games, it is possible to detect a pattern; in others, the sequences are random. The pairs of pupils were asked to detect rules where there were any, and later they discussed the differences between the two kinds of sequence. The researcher warned them that apparent patterns could occur by chance in the non-rule-governed sequences, but these patterns did not hold over a larger sequence of trials.

The third session was devoted to two highly familiar ways of producing random sequences, throwing dice and tossing coins. The session began with the students throwing dice repeatedly and recording the results of each throw. Some pupils had a normal, and therefore a "fair", die: the others had a loaded die. The pupils were not told about this difference beforehand, but needed little time to detect it, and then they were asked to discuss the difference in predictability between the randomised and non-randomised sequences.

The fourth session began with the teacher asking the children to distinguish events that are quite unlikely but could happen from those that are quite impossible. This is a distinction which children normally find hard to make (Piaget & Inhelder, 1975; Shtulman, 2009; Shtulman & Carey, 2007) but which is highly relevant to the understanding of probability. The children were presented with pairs of hypothetical events or actions, such as ‘making an umbrella out of glass’ and ‘making an umbrella out of air’ and were asked to discuss whether each of these was possible or quite impossible. The session ended with a modified bingo game, in which digits were drawn separately to form a two-digit number and were not replaced. The children were asked to discuss the predictability of the next number to be called, considering that some numbers became impossible.

The final session began with an activity that involved drawing marbles of two different colours from a bag. The important distinction in this activity was whether each marble that was drawn was then replaced or not. If the drawn marbles were replaced, the status quo was restored each time and the probability of picking each of the colours was as it was when the activity started, but when the marbles were not replaced the relative probability of the colour of the next marble changed with each draw. The aim of this activity was to draw the children’s attention, through discussion, to the fact that the composition of the contents of the bag is all that matters. The session ended with the children throwing dice and making predictions for the result of each throw. In one game the children threw a single die and in another they three two dice at once, in which case their prediction was about the sum of the pair of dice. The point of comparing these games was that the range of possibilities was different in the two cases, and also that the possibilities were equiprobable when one die was thrown, but not equiprobable when two dice were thrown (e.g. you are much more likely to throw a total of 7 than a total of 2 or of 12). The children would subsequently be introduced to problems about throwing two dice in order to work on understanding the sample space; at this stage in the programme, the focus was simply on the fact that only some outcomes were possible for the sum and that some totals were more likely than others.

Assessments

All the assessments were given as group tests. Each child had a booklet with one item per page and entered the answer on the same page. All of the items were multiple choice. Six were designed to test children’s discrimination between random and determined outcomes: e.g. counters black on one side and white on the other, thrown up in the air (randomised) in one item or turned over to the other side (determined) in another; a pack of cards arranged in suits and then shuffled in another item. The remaining three were a replication of Chiesi & Primi's (2009) study about the independence of successive events in a random sequence after a run of one particular outcome: e.g. one item was about a child drawing balls from a bag which contained an equal amount of red and green balls: the fictional child draws a ball from the bag and then replaces it four times and, on each of the four draws, the ball is red. The question is about what is likely to happen on the fifth draw. Is it more likely that the child will draw a red ball than a green one, or more likely that he will draw a green ball, or are these two outcomes equally likely? The pre-test and immediate post-test were identical, but the delayed post-test did not include items about determined sequences. In one item the number of balls in the two colours in the sack was equal: in the other two items the sack held more balls of one colour than of the other.

The children were also asked to write a justification of their choices in two of the items (one in which the balls‘ colours were equal and the other in which they were unequal) about the independence of events in a random sequence. The justification was given a point for each relevant aspect mentioned. The basis for coding these justifications was the assumption that five possible relevant aspects to each of the items could be mentioned in the children’s justifications. These were:

- Uncertainty of which colour would be drawn. *“you don’t know what will happen it is just luck”*

- Randomness: the random arrangement of the balls in the sack. *"because the marbles are mixed up*"
- Initial relation between two quantities (equal/unequal). *"it is because there is the same number of colours" "because there are more green marbles than red"*
- Replacement: the fact that replacing the drawn ball after each draw kept the relation between the two colours constant. *"because she put the marbles back in the bag"*
- Independence of successive events: what happened in the first four draws has no effect on the outcome of the fifth draw. *"it may have been just luck 4 purple marbles"*

Thus a justification mentioning that what happened in the first four draws has no effect on the outcome of the fifth draw, and that replacing the drawn ball after each draw kept the relation between the two colours in the bag constant, and that the four successive draws of the same colour were independent of each other, was given 3 points) (Inter-rater reliability in pre-test = .97).

RESULTS

Pre-intervention Test Scores

The main feature of the children's scores in the initial pre-test, which they took just before the interventions began, was that they were very good and much better than we had expected given the reults of previous research on children's understanding of randomness. The number of correct choices in the three randomised/unpredictable items_and in the three determined/predictable outcome items was extremely high. Eighty-two per cent of the children's choices in these 6 items were correct: 89.67% of the children's choices were correct in the randomised items and 74.33% in the determined items. These scores suggest that children of this age do understand the consequences of randomising quite well.

The children's scores for the three items that dealt with the independence of successive events in a random sequence were also surprisingly good. In the item in which the sack held an equal number of balls of the two different colours, 74.7% of the children made the correct choice: only 17.3% of the children made positive recency and only 8.0% made negative rcency errors. The number of correct choices in the two items in which the numbers of balls in the two colours were unequal were also high: 67.5% correct choices when the first 4 draws were of the minority colour and 76.0% correct choices when the first four draws were of the majority colour. These high scores present a very different picture from the one given in the Chiesi & Primi study. Recall, for example, that only 40% of the 10 year olds in the Chiesi & Primi study made the correct choice in the item in which the sack contained an equal number of blue and green balls: recall also that the college students in their study scored no better than the 10-year old children.

However, the justifications that the children gave for their choices in the two items in which we asked them to justify their choices were terse and limited: none ever touched on all five possible relevant aspects. The independence of successive events (or the irrelevance of previous events) was only given as a justification by 5.2% of the sample in the first of the items and by 4.1% in the second. The commonest relevant aspect mentioned by the children was the relative number of the two colours in the bag, mentioned by 35.5% of the sample in the first of these two items (equal numbers of each colour) and by 67.1% in the second item (unequal numbers).

Post-intervention Test Scores

The correct choices made by the Randomness group increased more from pre- to post-test than those of the other two groups in the randomness vs. determined items (Table 1). In the independence of successive events items (Table 2), the improvement in the Randomness group was also greater than in the other two groups. Because of the generally high pre-test scores the differences between groups were modest.

	Pre-test scores		Immediate post-test scores	
Groups	Randomised	Determined	Randomised	Determined
Randomness	2.63	2.38	2.96	2.63
Directed numbers	2.68	2.18	2.86	2.55
Unseen Control	2.76	2.14	2.60	2.10

Table 1

The mean number (out of 3) of correct choices by children in the three groups in the randomised and determined items in the pre-test and the immediate post-test

Group	Pre-test	Immediate post-test	Delayed post-test
Randomness	2.38	2.88	2.79
Directed numbers	2.05	2.45	2.41
Unseen Control	2.14	2.54	2.50

Table 2

Mean number (out of 3) of correct choices in the three items on the independence of successive outcomes in a random sequence in the pre-test, the immediate post-test and the delayed post-test

An analysis of co-variance of a combined score for correct choices in all the randomness items in the Immediate and Delayed post-tests (the covariate was the equivalent pre-test score) established that the pupils given the Randomness intervention had improved significantly more than the pupils in the other two groups (p=.019).

The improvement from pre- to immediate post-tests in the relevance of the justifications (Table 3) was also significantly greater in the Randomness group than the other groups (p=.004).

Groups	Pre-test justification scores	Immediate post-test justification scores	Delayed post-test justification scores
Randomness Intervention	1.83	2.71	2.04
Directed numbers Intervention	1.68	2.05	1.50
Unseen Control	1.24	1.71	1.68

Table 3

The mean number of relevant aspects mentioned in the children's justifications.

These results suggest that there exists among 9 and 10 year old children a widespread, but probably implicit, understanding of basic aspects of randomness, which can be improved and can be made more explicit by teaching that involves active experience and discussion of random arrays and sequences and of the effects of randomisation.

DISCUSSION

The two main aims of the project were (1) to find out in the pre-test what 9 & 10 year old children understand about randomness and the effects of randomisation, and (2) to see whether it is possible to teach children about randomness.

The answer to the first question was generally positive. The results of the pre-test assessments suggest quite strongly that at this age children who have never been formerly taught about probability nevertheless do have quite a lot of informal knowledge about the relation between randomness and uncertainty. Most of them could discriminate quite well between outcomes that were certain because they were determined and outcomes that were uncertain because of randomness, and most children also showed a good understanding of the independence of successive events in a random sequence.

Both these conclusions about primary school children's untutored knowledge about randomness and randomisation are to some extent at odds with the results of previous research, but the difference between the results of previous research on children's understanding of the independence of successive events and our own findings is particularly striking. The items that we used to measure the extent of this understanding were a direct replication of those devised by Chiesi and Primi, and yet the children in our study made far more correct choices and far less positive or negative recency errors than any of the groups in the Chiesi-Primi study including their adult group of students. The only common pattern in our results and theirs was that the 9- to 10-year old children in our study and the 10-year old children in their study made more positive than negative recency errors in both studies, but the much small number of errors of either type in our study than in Chiesi & Primi's diminishes the significance of this similarity.

It is hard to explain this radical discrepancy between the two studies. One possible reason could be that we stressed very clearly, and perhaps more clearly than Chiesi and Primi, that the character who made the first four draws from the bag always put the marble he had just drawn back in the bag before the next draw, and we put an animation into our PowerPoint slides that represented this replacement. However, a failure to grasp that the marbles, all of the same colour, picked in the first four trials were immediately replaced, could only lead to negative recency errors, since the person not noticing the replacement would think that the number of marbles of this colour in the bag, had diminished by four and therefore would be less likely to be drawn again . So this would not explain why the 10-year olds in the Chiesi & Primi (2009) study made more positive recency errors than the children in our study and more positive than negative recency errors in general.

At any rate we are more confident in our results than in those of Chiesi & Primi (2009) because the children in the two groups who were not instructed about randomness (the Directed Numbers Group & the Unseen Control Group) maintained the high level of correct answers about the independence of successive events in the immediate and delayed post-tests as well as in the pre-test, which shows an impressive reliability in the successful way that they dealt with questions about successive events (Table 2). We conclude that 9 & 10 year old children have a much better understanding of the independence of outcomes in a random sequence than previous research has claimed.

Turning to the second question, which is about teaching randomness to primary school children, our results do suggest that one can teach them successfully about aspects of randomness. This is useful to know since very little research has been done in the past on teaching children about randomisation and uncertainty. Our results suggest that the teaching given to the Randomness group did improve their understanding and reasoning about randomness to an encouraging extent, though not as strongly as we had predicted. The reason for the modest improvement in the Randomness group's scores was almost certainly that there was not much room for improvement: the children's performance in the pre-test before the interventions began was remarkably high already.

Nevertheless the improvment in the Randomness group's scores after being taught about randomness was significantly greater than in the other two groups. The intervention had a similar effect on the level of justifications given by the children for their choices in two of the independence of successive events items. The greater improvement in the quality of these justifications by the children in the Randomness Intervention group than by the other children (Table 3) is clear evidence that teaching children about randomness helps them to reason about uncertain events. This difference, taken together with the generally high level of correct choices made by the children in all three groups in the pre-test, suggests that children acquire an effective but implicit understanding of the effects of randomisation by the age of 9 to 10 years, and that one of the main effects of instruction about randomness is to make this knowledge more explicit and therefore more open to discussion.

The Randomness intervention programme took the form of pairs of children solving problems that usually involved concrete material such as dice and cards or material presented on PowerPoint and then presenting their solutions to the rest of the group for discussions at the end of each session. This seemed to us to be a congenial as well as an effective way of improving children's understanding of randomness and of making this understanding more explicit. In other publications we will show that the same general approach to teaching helped the children in the Directed Numbers group to solve directed number problems to the much the same extent. We cannot say whether other teaching approaches would also lead to improvements in children's understanding of randomness (or of directed numbers), but this study has established that 9 to 10 year old children are ready for instruction about randomness and that the instruction that we devised does improve their understanding of randomisation and the independence of successive events in a random sequence. We believe that ours is the first intervention study to have shown this.

References

Batanero, C., & Serrano, L. (1999). The meaning of randomness for secondary school students. . *Journal for Research in Mathematics Education, 30*, 558-567.

Chiesi, F., & Primi, C. (2009). Recency effects in primary-age children and college students. *International Electronic Journal of Mathematics Education, 4*(3), 259-274.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: Reidel.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? *Educational Studies in Mathematics, 15*, 1-24.

Piaget, J., & Inhelder, B. (1975). *The Origin of the Idea of Chance in Children*. London: Routledge and Kegan Paul.

Shtulman, A. (2009). The development of possibility judgement within and across domains. *Cognitive Development, 24*, 293-309.

Shtulman, A., & Carey, S. (2007). Impossible or improbable? How children reason about the possibility of extraordinary claims. *Child Development, 78*, 1015-1032.

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