13th International Congress on Mathematical Education
Hamburg, 24-31 July 2016

INTERACTIVE EXPERIMENTATION IN PROBABILITY - OPPORTUNITIES, CHALLENGES AND NEEDS OF RESEARCH

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In this study we investigate interactive experimentation in probability teaching, in relation to the bi-directional relationship of theoretical and empirical probability. By adopting to Inferentialism we explore opportunities and challenges of interactive experimentation in probability to create a classroom practice that is characterized by a language game rich in giving and asking for reasons. Data is gathered from a lesson sequence with two classes experimenting with unknown and known sample spaces. The analysis reveals how interactive experimentation in probability can create engagement, elicit deterministic reasons and prepare for purpose and utility. It was also evident how the design idea of "Comparing Samples of Growing Size" could be used to empower interactive experimentation in probability by encouraging the students to zoom in their discussions on the bi-directional relationship between theoretical and empirical probability.

INTRODUCTION

In this paper we investigate interactive experimentation in probability teaching. We develop and analyze a teaching experiment on the bi-directional relationship between theoretical and empirical probability. Interaction is conceptualized in the frame of the social, pragmatic and inferential semantic theory of Inferentialism (Brandom, 1994).

Theoretical probability is based on the assumption of an equiprobable sample space and this is an assumption that can almost only be made in games of chance (Batanero, Henry, & Parzysz, 2005). To act as conscious citizens in a modern society, we need to be able to express an opinion on the outcome of random events based on the observed frequencies of the event. However, there are also times when we must be able to express an opinion on the likelihood of an event when it is practically impossible to produce empirical evidence of the frequencies of a random event. We must be able to make predictions based on the construction of the random phenomena, although we cannot make assumptions about equiprobability. This dual nature of probability modeling supports a view of probability teaching that aims to develop an understanding of the bi-directional relationship between theoretical and empirical probability (Nilsson, 2009; Stohl & Tarr, 2002).

Previous research on the learning of probability has been student-oriented. Central to the tradition has been to distil students' prior knowledge and intuitive ideas about probability in the form of framework or profiles (Jones & Thornton, 2005). The importance of this research tradition cannot be overstated. Frameworks of students' mathematics provide tools that can be used for specifying the starting points of a learning trajectory (Simon, 1995) and for anticipating student responses in a learning activity. However, frameworks of students' mathematics provide limited information on the social and situational nature of meaning-making; on how and why students express an understanding of a content, given social and situational circumstances (Nilsson, 2009). Starting of from this perceived limitation, the present study aims contributing with further insights on opportunities and challenges in using experimentation as a motor for whole-class discussions on the relationship between theoretical and experimental probability.

To prepare for the empirical phase of the study we proceed with a brief review of research, which highlights issues on the relationship between empirical and theoretical probability in experimentation activities.

PREVIOUS RESEARCH

Reviewing research containing experimentation with data Nilsson (2014) distinguishes two overall methodological and analytical directions regarding the bi-directional relationship between theoretical and empirical probability. One direction was the *mapping direction*. This is the case when researchers investigate students' understanding in situations where the theoretical model of a random generator is known from sample space considerations. The students are asked to reflect on how this theoretical model is mapped in data. The other direction is the *inference direction*. This is the case when students are asked to infer a theoretical probability from data, since information about the sample space is unavailable, limited or too complex.

Up to date, educational research on the bi-directional relationship between theoretical and empirical probability can be described as characterizing research. Researchers have tried to characterize state of affairs in relation to how students express understanding of basic ideas. In research, structured in accordance to the mapping direction, three basic ideas have been particularly focused. These are students' ability to *identify the sample spac*e, their understanding of *the role of the sample space in the distribution of data* and of *the role of the size of the sample* in the relationship between theoretical and empirical probability (e.g., Horvath & Lehrer, 1998; Nilsson, 2007, 2009; Pratt, 2000; Stohl & Tarr, 2002).

Probability education research according to the inference direction is not as extensive or developed as research according to the mapping direction (Nilsson, 2014). One reasons for this may be that processes of inference are connected to statistics more than to probability. Talking about inferences in probability is talking about probability generalizations that go beyond the collected data (Pratt, Johnston-Wilder, Ainley, & Mason, 2008). It concerns making an inference from data to the features of the random system producing the data, and to the theoretical probability distribution the random system implies (Makar & Rubin, 2009). Research in this direction has mainly focused on two basic ideas in the relationship between theoretical and empirical probability. On the one hand, it has focused on why or why not students use data to infer on a probability distribution. In this situation it has been shown that students often seek other explanations than in data to explain how random outcomes occur (Makar & Rubin, 2009; Nilsson, 2014) On the other hand, it has focused on how students evaluate information, gathered from short and long data series respectively. In this second situation, researchers have focused on how students evaluate and understand the role of sample size, to infer on an underlying probability distribution (Pratt et al., 2008).

Research has focused on characterizing students' conceptions of basic ideas and how their conceptions many times differ from what would count as desirable conceptions. The social and situational nature of learning and understanding in probability has received limited attention. The teaching in the present study is interactive; it combines group-work and whole-class teaching. In experimentation-based group-work students solve a problem or inquiry through exploring, developing and asking relevant questions, investigating and making discoveries. Main purposes of inquiry-based group-work are to increase students' engagements, encourage them to develop purposeful tasks and discern the utility of mathematics (Ainley & Pratt, 2010). What been noted, however, is that collaborative inquiry-based teaching benefits from not being to open, handing over all responsibility to the students' own explorations. To be effective for the learning in stochastics, group work also need structure (Kalaian & Kasim, 2014). Alternating between an unknown and known sample space and *Comparing Samples of Growing Size* (CSGS) were two overall guiding principles in structuring the design of the teaching in the present study.

THEORETICAL BACKGROUND

We will use Inferentialism (Brandom, 1994) as our theoretical perspective to account for how students articulate understandings of chance and probability in interactive experimentation in probability. Inferentialism stresses the social and pragmatic nature of knowledge. The theory acknowledges Wittgenstein's (1968) key idea that knowledge is a construct of language use (Sellars, Rorty, & Brandom, 1997), shaped through our participation in language games. However, in Inferentialism not all kinds of language use and language games are equally important. Brandom (1994) introduces the *Game of Giving and Asking for Reasons* (GoGAR) as the primary language game of Inferentialism (Brandom, 2000). The analytical construct of GoGAR encapsulates two key ideas of Inferentialism, the *space of reasons* and the *Social practice of a language game*.

What Inferentialism says is that our understanding moves within *the space of reasons*; we make up our mind about "what to think and do in the relevant domain in light of what there is most reason to think or do" (Bakhurst, 2011, p. 136). The space of reasons refers to our cognitive capacity of being able to classify and label information and reflect on how and why we respond to information as we do (Brandom, 1994, 2002). Understanding involves having a grasp of reasons, driven by the peculiar force of striving for more acceptable and better reasons (Bakhurst, 2011). Things mean something to us, i.e., have content for us, because we are responsive to reasons for understanding them in one way rather than another. Brandom takes *inferences* as the primary moves of how content travels in the space of reasons. An inferential move of content means that the content can both serve as and stand in need of reasons. Grasping the meaning of an expression (understanding its content) implies grasping the inferential relations the expression is involved in or used to articulate.

The space of reasons is essentially socially articulated. Rather than considering the space in which thought moves, Brandom suggests looking at the space of reasons from the perspective of language use (Sellars et al., 1997). On this account, Brandom adheres to the pragmatic perspective of the late Wittgenstein (1968) and his principle idea that being a meaning-maker involves being a player of a language game. For Brandom, however, not all kinds of language use and language games are equally important. Connected to our cognitive capacity of developing space of reasons, Brandom stresses the Game of Giving and Asking for Reasons: being a player, a meaning-maker, in the social, pragmatic practice of a language game involves being able to give and ask for reasons of how and why things happen as they do.

METHOD

We analyze data gathered from a small-scale teaching experiment with students from year 5 and 6 in a Swedish elementary school. The aim of the task was to engage students in GoGARs involving the bi-directional relationship between theoretical and empirical probability.

Setting the stage of the teaching experiment

The teaching involved two classes, one Grade 5 and one Grade 6. Two researchers (the authors of the paper) and the two class teachers constituted the developmental team of the teaching. There were six lessons conducted in the Grade 5 and five lessons conducted in the Grade 6. The lessons in the Grade 5 preceded the lessons in the Grade 6. Between the lessons the team met for reflecting on a previous lesson and for planning the next lesson. Hence, the lessons of the Grade 6 drew on what we learned from the proceeding lessons of the Grade 5. All lessons were video-recorded. Due to space limitations we will focus on what happened in the Grade 5.

The overall instructional design of the teaching originated from Brousseau, Brousseau, and Warfield (2001). A bottle containing a number of small colored marbles was used as a random device. In the first lesson the bottle was covered. Neither the students nor the teachers knew the content of the bottles. When turning the bottle, the color of one ball was revealed while remaining inside the bottle. By this construction we created a constant, unknown sample space. The activity was presented as a competitive race in the first lesson with three contestants (blue, white or red marble) on a six-step track. As one of the three colors where observed on a bottle turn, that color advanced one step down the track. The students where asked to predict which color would first get six observations during each race. Based on the topics discussed by the students in the first lessons, the upcoming three lessons revolved around a transparent bottle; a bottle with a visible sample space.

When working with the transparent bottle the students were challenged to reflect on the role of sample size to match the content of the transparent bottle. The students returned to the covered bottle in the last lesson. In the last lesson, the students were supposed to use their developed understanding of the bi-directional relationship between sample space and empirical probability to infer on the content (the sample space) of the covered bottle.

RESULT AND ANALYSIS

Lesson 1

There are particularly three issues of opportunities and challenges that emerge in lesson 1. These are *creating engagement, elicitation of deterministic reasons* and *preparing for purpose and utility* (Ainley & Pratt, 2010). The class is playing with the covered bottle and it is the teacher who is in charge of handling the bottle.

Creating engagement. We note three factors that particularly support students to be engaged in the race. The first factor relate to an interactive strategy by the teacher. During the first race, Karen walks around in the classroom and let as many students as possible be engaged in seeing and telling which color appears when turning the bottle. The second factor speaks to the competitive nature of the experimentation. Rich learning activities in probability and statistics build on a question that serve the means of a driving motor in experimentation (Makar & Rubin, 2009). In the present case, questions like "What color wins?" and "Why did that color win? Serve the means of such driving questions. The third factor draws on the second in that the game element provided the teacher certain opportunities to interact with the students to encourage their engagement and curiosity. This is what is happening when the teacher cares for red, the only color that did not appear during the first game. Before the second race the teacher asks the students to guess which color they think will win. Five students guess white, 13 blue and four red. The teacher's vote is included in the four votes for red. After two blue in the second round the teacher expresses "Come on now red." We have no reasons to believe that the teacher's vote and care for red is an expression of the gambler's fallacy (Fischbein, 1975). Instead, we interpret her act as an act of creating engagement and a joyful spirit in the activity. She repeats this act during the play and we can see how it supports students' motivation for the game. The teacher's focus on red also shows the social nature of learning. When the students are asked to predict the third game, nine students choose red as the winning color. This despite the fact that they only observed one red from the two first game and, after the second game, several students have expressed that blue probably win because they think there are more blue in the bottle. We claim that voting for red should not be considered to be a cognitive limitation. We claim that the reason for why students predict red as the winning color in the third race is to be the result of social structures, established by the teacher's focus on and expressed concern for the red color.

Elicitation of deterministic (material) reasons. After the second game the class develops a GoGAR on the proportions of marbles in the bottle. For instance, several students expressed that the reason for the many blue observations is because they think there are more blue marbles in the bottle. However, after the third round another GoGAR appears. This time the GoGAR involves material reasons. Focusing on material reasons was even more prominent in the other class. Examples of material-oriented GoGARs were when students tried to explain the result of the game by referring to how the bottle was turned, to the speed in which the bottle was turn, to the order in which the different colors had been inserted into the bottle and to an idea of differences in weight between the marbles. What we see when reflecting on the appearance of the two different GoGARs we see that there is a social aspect involved. What we see is that the first student contribution has a major impact on shaping subsequent contributions. In the first occasion, the first contribution concerned the proportions of colors in the bottle. Even though the next contributions did not exactly acknowledged the same proportions as the first, all concerned the proportions of the different colors in the bottle. A similar social pattern appeared in the grounding of the material-oriented GoGAR. The first student says, "it might be that there is a magnet in the bottle [the rest is a bit unclear but wordings imply that he means that marbles can get stick to the magnet, hindering the marble to drop down]". This suggestion then invites for a range of ideas that refers to the construction of the bottle, the features of the marbles and the turning of the bottle.

Preparing for purpose and utility. Based on our expectations prior to the lesson there is a lack of probabilistic content in the GoGARs emerging in the first lesson. Focus is on creating engagement and a joyful atmosphere. The class does not engage in any longer or probing talks about the underlying structure of the race. However, the lesson should not be viewed in isolation. What our analysis reveals is that experimentation-based teaching needs to take time. We need to view the first lesson as a lesson of a teaching sequence in which the class is encouraged to dig deeper into content in subsequent lessons. It is from such a holistic teaching perspective we consider the first lesson crucial in preparing for and eliciting aspects of purpose and utility (Ainley & Pratt, 2010). In the first lesson the teacher captured the students' interest and evoked their curiosity. She could take advantage of such engagement in subsequent lessons, and zoom in on and maintain pupils' interest to understand the regularities of the bi-directional relationship between theoretical and empirical probability (Stohl & Tarr, 2002).

Lessons 2, 3 and 4

Making the regulation of the random experiment transparent. Although occasions appeared during the first lesson where students articulated an awareness of the role of the sample size as an explanation for the distribution of outcomes, it also became clear how difficult it was for both teachers to turn the GoGARs away from deterministic issues. In particular, none of the two classes came to the conclusion that they could find out the content of the bottle from frequency information. In an effort to make the relationship between sample space and frequencies transparent to the students it was therefore decided to let the students experiment with transparent bottles during the following three lessons. This decision was based on the assumption that, with a transparent bottle the students would see that there is nothing dubious with the bottle. We wanted to make clear that what regulates the distribution of observed marbles in our trials is the proportion of marbles in the bottle.

Comparing Samples of Growing Size (CSGS). The design of lessons 2, 3, and 4 was based on an idea of *CSGS.* The class was divided in seven groups. All groups interacted with a bottle, containing the same uniform distribution of marbles. First each group compared two samples of 25 observations. Then, the groups combined these two samples in a sample of 50 observations. Each group ordered their 50 observations in bars on a paper sheet. They cut out each bar and nested them in the form of one large sample at the board at the front of the classroom. The board showed one white, one blue and one red bar. This visualization provided the teacher support in developing a discussion on relative frequencies: the relative frequencies of the class' 350 observations matched the uniformity of the bottle sample better than most groups' results. Moreover, the teacher was able to highlight the necessity of connecting the content of the bottle to the proportions of the different colors in the bottle instead of judging on the absolute frequencies.

Lessons 5 and 6

In the last two lessons Karen's class is supposed to use their understanding, developed in the proceeding three lessons, to infer on the content of the covered bottle. At the beginning, however, it is still not clear to the students how to do this. The teacher needs to take a rather active role in turning the students towards empirical probability; to the use of samples and connecting relative frequencies of a sample to the proportions of colors in the bottle. The same seven groups as before conducted a series of 50 observations from the covered bottle and were asked to infer on the content of the bottle from these 50 observations. Then, the teacher put all 350 observations together on the board. The empirical probability (relative frequency) of each color resulted in P(white) $\approx$0.39, P(blue) $\approx$0.48 and P(red)$\approx$0.13. Based on these results, the class discussed different possible proportions of the colors in the bottle. The situation provided Karen many opportunities to develop rich GoGARs in the class about the relationship between the calculated frequencies and the content of the bottle. Particularly, Karen could use the situation to move the GoGAR forward by the many opportunities she was provided to ask students for reasons about how they inferred on the content of the bottle from the frequencies. Moreover, that the bottle was covered and that the class was forbidden to open the bottle became crucial in promoting rich GoGARs in relation to the bi-directional relationship between theoretical and empirical probability (Stohl & Tarr, 2002).

DISCUSSION AND IMPLICATIONS FOR FUTURE RESEARCH

Taking an inferentialist perspective (Brandom, 2000) on interactive experimentation in probability teaching, the analysis highlights how responsive humans are to reasons and, particularly, to reasons of a deterministic kinds (cf., Konold, 1989; Nilsson, 2013). However, the solution is not that teaching should avoid experimentation because students being responsive to material and physical reasons instead of data-grounded reasons. Instead, teaching should confront students' deterministic commitments and have them experience why it would be more powerful to commit to a probabilistic and statistic perspective instead. Teaching in mathematics should provide students the opportunity to experience the utility of mathematical ideas (Ainley & Pratt, 2010). In experimentations, students' deterministic commitments can be made explicit and be the target of reflection and evaluation. As been disclosed in the present study, in experimentation, students can develop purposeful tasks (Ainley & Pratt, 2010) such as formulating hypothesis, make predictions about the outcomes of the experiment and test their hypotheses and predictions. Such processes may confront students with the shortcomings of determinism and help them understand the power of approaching a random phenomenon from a stochastic perspective; a perspective that takes into consideration random variation and is based on sample space composition and frequencies.

In order to exploit the potential of experimentation-based, interactive teaching needs to be structured (Kalaian & Kasim, 2014). Alternating between an unknown and a known sample space and *Comparing Samples of Growing Size* (CSGS) were two overall guiding principles in structuring the design of the teaching in the present study. The covered bottle was important in engaging the students in the activity. It became a mystery. It contained a secret! This secret supported students' engagement and helped the teacher to push and move the GoGARs forward (Brousseau et al., 2001). Another important issue in the teaching of mathematics that the covered bottle made visible speaks to socio-mathematical norms (Yackel & Cobb, 1996). In their ordinary teaching the students were used to confirm a mathematical solution by the answer or solution provided at the back of a textbook and, when this was not possible, the students, and the teachers', became frustrated and had no readiness for how they would handle the situation and rely on evidence other than purely positivist evidence. The CSGS empowered the experimentation activity in three respects. Firstly, it supported the elicitation of GoGARs on the unpredictability of random behavior in the short run and its predictability in the long run. Second, it provided a context for discussions on why we need to focus on proportion, on relative frequencies, when inferring on the composition of sample space. Thirdly, the systematic growth and comparison of samples provided the teacher certain opportunities to zoom the GoGARs in on the role of sample size in the connection between sample space and relative frequencies.

The present study shows the complexity in opportunities and challenges of experimentation in the teaching of probability. It shows the need to take a fine-grained account of the social and situational nature of how students express and develop an understanding of randomness and probability in such learning environments. On this account, we invite future research to develop methodologies for making such accounts and for developing conceptual frameworks guiding the design and analysis of rich interactive experimentation in probability.

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