Betting as a Pathway to the Law of Large Numbers – Self-construction of Strategies for Initiating Conceptual Change

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Summary

Our design-research study starts from the assumption that children should gain experiences with the phenomenon of randomness and the law of large numbers long before a systematic introduction to formal probability. That is why we developed a learning environment (Hußmann & Prediger, in press) which approached randomness by betting – experimenting – reflecting. This contribution gives short insights in an ongoing design experiment study that deals with challenges and chances when children shall discover the law of large numbers by constructing their own good and more secure strategies for betting. This contribution concentrates on one aspect out of the larger study, namely on the role of constructing and changing strategies as important element of the pathway to understanding the law of large numbers.

Theoretical Background of the Design Decisions

Learning probability is well known to pose obstacles when children bring in their everyday experiences and prior conceptions which are not always in line with the mathematical concepts concerning chance and probability. Already described by Fischbein's (1975) notions primary versus secondary intuitions, Shaughnessy (1992) calls this phenomenon a "major problem for the teaching and learning of probability and statistics concepts" (p. 472). Since then, a wide range of individual prior conceptions concerning many different mathematical aspects have been documented in empirical studies (see Shaughnessy, 1992 and Borovcnik & Peard, 1996 for overviews).

Conceptual change in horizontal and vertical view

The importance of individual prior conceptions in all domains is explainable in constructivist terms: individual, active constructions of mental structures always build upon the existing prior mental structures by accommodation to experiences with new phenomena (Cobb, 1994). In parallel with this general constructivist account for the crucial role of prior conceptions, the so-called *conceptual change approach* was developed (Posner et al., 1982; Vosniadou & Verschaffel, 2004) according to which learning mainly means "re-learning, since prior conceptions and scientific conceptions are often opposed to each other in central aspects" (Duit & von Rhöneck, 1996, p.158, translation SP). Konold (1991) first used the conceptual change approach in probability for describing students' difficulties and possible strategies to facilitate their conceptual change challenges. He suggested several teaching techniques for confronting and overcoming stochastic misconceptions. This is in line with classical propositions given by Fischbein (1982) for building secondary intuitions onto the partly problematic primary intuitions:

> "For instance, in order to create new correct probabilistic intuitions the learner must be actively involved in a process of performing chance experiments, of guessing outcomes and evaluating chances, of confronting individual and mass results *a priori* calculated predictions, etc." (p. 12)

Since then, many other authors, with or without explicit reference to conceptual change, developed learning environments that aimed at a sustainable conceptual development from prior individual conceptions to the intended mathematical conceptions by generating cognitive conflicts (e.g. Aspinwall & Tarr, 2001). The guiding question in this *vertical view* on conceptual change is the following one: How can individual prior conceptions be overcome and further developed to appropriate ones?

This early approach of conceptual change was guided by the important constructivist idea to take students' intuitions into account and not to teach as if they were "empty sheets". Nevertheless, the far reaching aim of "overcoming" individual prior conceptions by mathematics classrooms is critical. First, it is not always realistic, as individual conceptions often continue to exist next to the new conceptions, being activated situatively (cf. in general Duit & Treagust, 2003; Smith et al. 1993; for probability e.g. Shaughnessy, 1992; Konold, 1989). Second, rather than a substitution of prior conceptions, the more adequate aim for a conceptual change process in constructivist terms is the *shift of contexts* in which everyday and scientific conceptions are to be activated.

> "Conceptual change does not imply that initial conceptions are "extinguished". Initial conceptions, especially those that hold explanatory power in nonscientific contexts, may be held concurrently with new conceptions. Successful students learn to utilize different conceptions in appropriate contexts." (Tyson et al., 1997, p. 402)

In Prediger (2008) this perspective on conceptual change as based on accepting the persisting co-existence of prior and mathematical conceptions is called a *horizontal view* on conceptual change. It is in line with Abrahamson and Wilensky's (2007) approach to take primary intuitions as legitimate ideas which can persist when weaved into a new framework. But in contrast to Abrahamson and Wilensky, we do not believe that all building blocks for normative mathematical reasoning are already in every student's repertory. Instead, we start from the idea that the repertoire of experiences has to be extended as most students do not yet dispose of all necessary conceptions and experiences. On this base, we change the design question for conceptual change into the following: How can learning environments support the extension of individual repertoires of prior experiences and conceptions, and how can learners be enabled to choose the adequate conceptions in varying contexts?

Short-term and Long-term Perspectives as Background for Context-adequate Choices

In previous empirical studies (Prediger, 2008), we searched for conditions for children to choose adequate conceptions. In qualitative experiments with game situations where children (about ten years old) were asked to guess the sum of two dice, we could reproduce Konold's (1989) observation that many individual difficulties with decisions on uncertainty arise not only from problematic judgements on probability, but root deeper in "a different understanding of the goal in reasoning under uncertainty" (Konold, 1989, p. 61). Konold could give empirical evidence for a predominant individual wish to predict single outcomes when dealing with instruments of chance, the so-called *outcome approach*. Similarly, the children in our experiments tried to explain the last outcome, drew conclusions from one outcome, tried to predict the next outcome, found evidence for the unpredictability of outcomes when outcomes did not follow the theoretical considerations etc. This outcome-approach was a main obstacle for the children to activate suitable probabilistic conceptions. But unlike Konold's results, our study could not reproduce the outcome approach as a *stable* phenomenon describing the behaviour of some children, but as *appearing situatively*: They switched between the perspectives even without being aware of it: the same children could adopt a long-term perspective two minutes later, e.g. when considering a tally sheet with 200 outcomes.

To make a point: We consider the so-called "misconception" of the "Law of small numbers" (see Tversky & Kahnemann, 1971) not to be wrong per se but only used with an inadequate domain of app lication. Hence, the well-known divergence of short-term and long-term perspectives turns out to be the *crucial background for context-adequate choices*.

Here, we see connections to Abrahamson and Wilensky's (2007) dialectic idea of bridging tools and learning axes: From a mathematical point of view, the law of large numbers tells us that probabilistic predictions are more reliable in the long run (Langrall & Mooney, 2005). Consistent with this point being crucial for understanding the meaning of probability judgements, Moore (1990) puts this aspect into the centre of his introduction to probability and turns this characteristic of randomness into a definition of the scope of probability.

> "Phenomena having uncertain individual outcomes but a regular pattern of outcomes in many repetitions are called *random*. "Random" is not a synonym for "haphazard" but a description of a kind of order different from the deterministic one that is popularly associated with science and mathematics. Probability is the branch of mathematics that describes randomness." (p. 98)

Hence, the shift from a short-term focused outcome-approach to a long-term perspective on randomness is one crucial challenge for conceptual change. The horizontal view on conceptual change emphasizes the need for individuals to become aware of the law of large numbers as a precondition for predicting outcomes in long-term random situations by probabilistic terms. The construction of our learning environment was guided by the idea to allow children to recognize the stability of long run frequencies and to make clear that the regularity described by probability does not apply to short sequences of random outcomes (Moore, 1990, pp. 120-121; see also Konold, 1989; Prediger, 2008).

The learning Environment in the Design Experiment

Taking explicitly into consideration the individual repertoires of prior conceptions based on the outcome approach, a learning environment was designed that intended to complement and hence enrich them with mathematical conceptions of probability in a prognostic sense. That means probability was introduced as a good predictor for relative frequencies for a large number of throws of a die (Riemer, 1991). This conception is an easier variant of a frequentistic concept and combines theoretical and empirical aspects. The explicit reflection on effects of the law of large numbers in the learning environment is aimed at enabling students to activate adequate conceptions in the right situation, i.e. to facilitate conceptual change in a horizontal view.

The design followed Fischbein (1982) in his emphasis on the activity of guessing outcomes of chance experiments. In line with other designs for learning environments that embed this activity into game situations (Amit & Jan, 2007; Aspinwall & Tarr, 2001), we situated our design experiment in a learning environment with the game "Betting King" for grades 5 and 6 (students aged 10 to 12 years) (see Hußmann & Prediger, in press). In the game, children bet on running coloured animals (like in a horse-race), being fuelled by throws of a coloured die with asymmetric colour distribution. The term *betting* was used deliberately to evoke a tension and seriousness for the guessing activity, even without any stakes. Winner of the game is the person with the best rate of correct bets in a series of laps. The need for students to develop strategies in order to win the game gives reason for investigating the occurring patterns. Due to the asymmetric colour distribution on the 20-sided icosahedron die (7 red, 5 green, 5 yellow, 3 blue), the red ant has an advantage. A good bet is hence to choose the red ant; this is quickly obvious for most children. In this point, the learning environment starts from existing individual resources, namely an ordinal intuition on probability, usually expressed by "most likely – least likely".

The learning goal of distinguishing short-term and long-term situations was strengthened by the option to choose the numbers of throws in the subsequent game. This extends the reflections: Varying the number of throws, which is a better choice to make the bet on the red ant as secure as possible? A computer simulation of the game helped to quickly identify a larger number of throws being more favourable. To sum up, the design of the learning environment was driven by the core idea that the self-construction of good and rather secure strategies might initiate conceptual change processes in a horizontal sense.

Research Questions and Design of the Study

The learning environment was used in a design experiment with four classes (grade 5/6, children of 10 to 12 years) with their own mathematics teacher. This paper only covers the analysis of the first lessons in one class. The design experiment is embedded in the larger long-term project KOSIMA, aiming at design research in grade 5-10, guided by B. Barzel, S. Hußmann, T. Leuders and S. Prediger (http://www.kosima.uni-dortmund.de). The second author, Katrin Rolka, was part of the research team.

The experiment was designed to analyze the processes of conceptual change in the classroom. Although our research concerns a whole range of aspects and manifested effects on the processes of conceptual change, this article focuses on the development of betting strategies in the first phase of the game in order to evaluate the role of self-construction of strategies. In detail, we studied the following questions:

- Which strategies do the students use at the beginning, and which prior conceptions can we reconstruct behind students' them?
- How do the students change and / or enrich their repertoire of strategies?
- How do students become aware of the conditions when their bets are most suitable?

These questions were treated by qualitative analysis of the data comprising students' written strategies, their journals written while reflecting and playing (in which we could find the enacted strategies), as well as the videotaped episodes from interaction processes during discussions in the whole class. A forth question can only be treated in a tentative outlook since it requires a more detailed analysis of transcripts from the interaction processes in the classroom:

- What might support the change of strategies and conceptions in the classroom situation?

Empirical Findings

Presenting a First Case: Steps in Maurice's Learning Pathways

With Maurice, we present the case of a student who took the intended learning pathway in a prototypic way. His case also serves us to introduce the course of the considered four lessons and some categories by which the three documented steps in the students' learning processes were analyzed and coded.

First step: In the first lesson, all students played the game in groups of four, first placing their bets and then letting the animals run. After playing several times, the teacher initiated a whole-class discussion on experiences with the game (see below). After this discussion, the students were asked to investigate the game in a more systematic way. Therefore, they recorded their bets and the actual outcomes into a table in which the numbers of throws were fixed for the first game: 1, 2, 5, 10, and 20. When the students were asked to formulate a good betting strategy at the end of the lesson, Maurice wrote down the following:

> "My strategy: I pick the one that somebody else has already picked. And I take my special throwing technique."

This initial statement includes two types of strategies: The first part shows a strategy that is based on social considerations. In a documented conversation, Maurice explicated that while copying other students' choices, one can never loose alone and never be the worst player in the group. The second part of his statement refers to a strategy that we coded as "influencing strategy" as he alludes to his special throwing technique. At this stage, Maurice makes no explicit reference to probabilistic considerations besides his prior conception that randomness can be influenced by good throwing. It is worth to note that he sticks to these strategies despite of the classroom discussion that took place before.

Second step: After playing the game again in the second lesson, students had the opportunity to reformulate their strategy. Maurice noted:

> "Always bet on ant because she is the champion in our group."

Maurice now changed the written and (as his journal on the bets in the game shows) enacted strategy and approached the mathematically desirable insight that the ant is more likely to be the fastest animal. Whereas other children extracted this insight from analyzing the (asymmetric) die, his justification of the strategy is rooted in an empirical view on probabilistic phenomena. We cannot say whether this change of strategies is rooted in further playing experiences or in the interaction between teacher and students that took place in the middle of the first lesson (see section on classroom interaction below).

Third step: In the third und forth lesson, students could experiment with varying numbers of throws on a computer in order to deepen their insight on context-specificity of the risk of the bet on the red ant. The results from all groups were brought together on an overhead projector and summarized in a table. Students were then invited to write a comment on the results. Maurice went beyond the task to comment on the results of investigation as he directly referred his observation to the betting strategies:

> "Always bet on ant because the die has more red spots (7), frog (5), snail (5), hedgehog (3). The ant doesn't always win because for small distances, the hedgehog, snail and frog could also win sometimes. And for longer distances, the others need luck."

This comment shows that Maurice now consolidated his strategy to bet on the animal that is most likely to win by theoretical considerations (the hint to the colour distribution on the die) and by empirical considerations (the analysis of the computer-experiments). Maurice could differentiate between short-term and long-term perspectives, saying that the bet on the ant is not secure for small "distances" (number of throws).

Maurice's learning pathway corresponds quite well to the intended learning pathway as his statements at different steps show how he began with his individual strategies, namely the social strategy (in which we cannot yet find manifest prior conceptions concerning randomness, it is driven by other rational considerations) and his influencing strategy (that corresponds to well documented prior conceptions on randomness as being manipulable, see Fischbein et al., 1991). During the work in the classroom, he extended his range of conceptions and strategies by adding an empirical view without completely abstaining from his initial strategies. Remark that his social strategy might be highly suitable for smaller numbers of throws.

The important point for the intended conceptual change process in a horizontal sense is the ability to differentiate when to use which betting strategy, namely the insight that the bet on the red ant is more secure for a larger number of throws.

Embedding the Case of Maurice into the Larger Picture of the Whole Class

The case of Maurice is typical of the investigated classroom in some aspects, but not in all. Table 1 gives an overview of different initial strategies that were documented by the 24 students (of which four gave no formulated strategy).

Initially written betting strategies	**Frequency**
Social strategy (look for other's choices)	3 x
Influencing strategy	5 x
Bet on red ant, only theoretically justified	5 x
Bet on red ant, only empirically justified	4 x
Bet on red ant, empirically and theoretically justified	4 x
Bet on red ant without reason	2 x
Bet on other animal, empirically based	3 x
No written betting strategy	4x

Table 1: Frequency of initial betting strategies (some kids expressed more than one strategy).

First step: Like Maurice, four other students resorted to influencing strategies of various forms, in which a non sustainable conception of randomness becomes manifest. It is an aim of the learning environment to provide enough experiences that contradict this prior conception of randomness.

In contrast, the social strategies, which emerged in three cases, are good examples of children's thinking being definitely rational in itself although not in line with stochastic learning goals. Hence, there is no reason to aim at "eliminating" them.

Although these strategies (co-)existed, most students (15 of 20 formulated written strategies) recognized that the red ant is a good bet already after a short time of playing. This confirmed the design idea that the learning environment can tie in with students' prior conceptions, namely with an ordinal intuition on "more likely" of which most students of this age dispose. The children varied their justification of this intuition.

> "Because there are more red stickers, I take the ant more often." (Marco)

"I take ant because it is often first." (Merve)

Like Marco, five students exclusively mentioned a connection between the distribution of colours on the die and the outcomes, hence referred to a theoretical justification. Like Merve, seven students based their judgement on empirical arguments, by referring to their experiences made while playing the game. Four students combined both resources of justification and hence demonstrated an already well-developed multi-faceted basic stochastic understanding. Only one student, Amelie referred to her empirical experience but came to another judgement, preferring the blue hedgehog (see below).

Second step: Although not all students wrote down new strategies in the second lesson (hence the data of the second step is incomplete), it is worth telling that only two students formulated diverging bets, all other written strategies focused on the red ant now.

"I always choose the snail, eventually she will win." (Neville)

"I choose frog, ant, snail because they win most often." (Tom)

Tom's strategy is in line with intended probabilistic conceptions as it is really the hedgehog which is the least likely to win. Hence, in the second lesson, Neville was the only student who formulated a strategy in which a divergent conception of the situation appears: The for the German language unusual future tense "eventually she *will win*" seems to be more inspired by an idea of variance than of probability.

Third step: Like Maurice, eleven out of 24 students in the class were able to formulate a distinction between small and large numbers of throws. The others needed more time in further lessons before they could express the difference between short-term and long-term perspectives in written texts.

What happened with the others, and what were the processes in between these steps? We will take two more snapshots.

The Case of Amelie – Obstacles and Conditions for the Learning Pathway

First step: In contrast to the above mentioned majority of students who already started with the bet on the red ant, Amelie is an interesting case because in the first meeting, she saw the blue hedgehog as the best animal:

"One has to bet on hedgehog because blue is thrown most often." (Amelie)

Like other students who based their reasoning on empirical considerations, Amelie also referred to her individual perceptions of empirical observations. Although her and her group members' bet records show that in the 20 laps played, the hedgehog won only four times, she declared the blue hedgehog as winner. This reconfirms the problem described by Borovcnik and Peard (1996): experiments in probability are not always likely to generate cognitive conflicts that support the construction of mathematically adequate conceptions. In our learning environment, the problem was reduced (but of course not eliminated) by the possibility of generating larger numbers of throws with a computer simulation and recording the results.

Second step: Amelie continued her learning pathway in the second meeting with the following remark:

"Sometimes, one can take ant but the ant can't win all the time." (Amelie)

Amelie seems to have learned that the red ant is most likely to win, but the linguistic markers "*sometimes, one* can take" give hints that she does not really feel comfortable with it – *somebody else* can take ant? Would she perhaps still prefer the hedgehog without the influence given by the classroom interaction? Her enacted bets correspond to her writings as she mostly (yet not always) bets on the red ant in the second lesson. We will analyze below one of the episodes of classroom interaction that sheds light on where Amelie's (fragile) knowledge that the red ant is the best bet might come from.

Third step: The not yet consolidated process of conceptual change is reflected by her inconsistent comments on the examined data in the third step.

> "On the table, it catches my eye that the ant has more points than the others. But the other players (snail, frog, and hedgehog) also won sometimes. The ant is not the fastest because the others also won sometimes.

Amelie correctly described the table and also mentioned that the ant does not win all the time. However, unlike Maurice, she was not explicit about the differentiation between the short-term and long-term perspective. Likewise, six other students did not succeed to explicitly formulate this distinction at this early stage but only later within the experiment. This shows how important it is to focus on this point in a learning environment.

Whole Class interaction as Important Context for Conceptual Change

What happened between the intermediate states of conceptual change as documented in the written utterances of the students? One important context for the individual processes of conceptual change is given by the moments of whole class interaction like the following episode in the middle of the first lesson. In this sequence, the teacher collected and valued the developed individual strategies:

Teacher:	Now, I would like to, a little bit of what you saw and noticed for the game and what was fun, that's what I would like to treat now (some students raise their hands).
Lance:	In our group, the (red) ant has shown up most often.
Teacher:	Maurice.
Maurice:	Andrew always bet on red, always won. Then I turned to red and I lost (students and teacher laugh). The red one was the champion here.
Teacher:	Faruk (Maurice continues to talk) Stop it, once again, and now, absolutely everybody is quiet. Faruk!
Faruk:	Actually, all of this is chance.
Teacher:	All of this is chance, mmh (points to Fred).
Fred:	What I liked most, there were, there were the ant, the frog and the snail all on 19.
Teacher:	Oh yes.
Fred:	And then, Michael brought mine forward, the frog.
Teacher:	Yes and, eh, Laurent just said something. What did Laurent say? I chanced to hear it (small pause), well, say it again (Laurent murmurs). Laurent, what did you say?
Laurent:	Nothing (teacher laughs).
Sara:	You said that it depends on the die.
Laurent:	There were always more reds on it than other colours.

Several students: Yes!

The episode shows the productive diversity of ideas and observations the students have built during their first playing experiences. Although the teacher does not explicitly judge some strategies or observations to be wrong, he subtly enforces some observations more than others and asks Laurent several times to share his insight into the asymmetry of the die as possible source for an advantage of the red ant. Laurent's observation is approved by several students. From this moment on, the theoretical justification of asymmetry of chances is a shared pattern of argumentation for a betting strategy in this classroom. By such kind of patterns of interaction, sociomathematical norms of what counts as important observation and as suitable strategy are established although the complete diversity of individual strategies is valued (Yackel & Cobb, 1996). Hence, the interactively established norms in this class were multi-faceted: On the hand, as the diversity of strategies was valued in the first lesson, the students recalled a variety of strategies, including e.g. social strategies, also at the beginning of the second lesson. On the other hand, only Neville wrote down a definitely diverging strategy. It seems that the students have learnt the lesson of what strategy counts most in this mathematics classroom.

Without being able to reconstruct for each student how exactly this episode influenced his or her self-construction of strategies and conceptions, we are convinced that this episode contributed to the phenomenon that only two children wrote something else than “red ant” as the best bet in the succeeding lesson (see above). At least for Amelie we suspect that the social context was, at first stance, more influencing than internal insights.

Conclusion and Outlook

This article intended to promote a horizontal view on students’ conceptual change challenges, namely the necessity to enrich the repertoire of individual prior conceptions with mathematical conceptions and to give students an orientation in which context to choose which conception. As this basic idea guided our design process, we put the law of large number into the centre of the learning environment while offering a bridge between prior and intended conceptions by starting with purely ordinal judgements (i.e. more or less likely).

The analysis of students’ conceptual development showed that many students already activate powerful strategies after playing the game only for a short time. That is why we are convinced that the game and the learning environment are suitable for integrating students’ prior conceptions on probability. After gaining more experiences in the learning environment, students are also able to extend and differentiate their repertoire of initial conceptions by reflecting on the risk of their bets depending on the number of throws. Indeed, the majority of them succeed to differentiate between smaller and larger numbers of throws quite early.

Nevertheless, we still would like to understand in more detail the development of the strategies and analyze more discussions about the strategies among the students in seat work phases. One reason for missing discussions might be that the students want to keep their strategies secret because they want to win the game. In order to find out more about the development of their strategies and the underlying probabilistic reasoning, it might be valuable to group the students in pairs of two first, asking them to work out a strategy for the game and then mixing the groups.

Like all empirical studies, also our study shows clear limitations in methodology and theoretical background. Obviously, considering steps in the process of conceptual change as documented in students’ journals can only give hints on intermediate steps. For understanding the learning processes in their course, also the small group discussions must be analyzed in order to understand the turning points in the processes of conceptual change.

Moreover, our analysis of whole class interaction brought us to the conviction that the learning process cannot be analysed by the theoretical perspective of conceptual change with its individual-psychological terms alone. In our further work, the analysis of the learning process must be complemented by a social perspective that gives better accounts for the emergence of social practices and taken as shared argumentations (Cobb et al., 2001).

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