

EFFECTIVENESS OF USE OF DIAGRAMS FOR TEACHING CONDITIONAL PROBABILITY FROM A SEMIOTICS VIEWPOINT

Ippo Ishibashi

Hiroshima University High School, Fukuyama
Graduate School of Education, Hiroshima University, Japan
ipishibashi@hiroshima-u.ac.jp

This study examines teaching approaches for learning conditional probability by using diagrams. Toward this end, we first identify conventional knowledge about diagrammatic representations of probability from a semiotics viewpoint. Next, we present effective diagrammatic expressions for understanding conditional probability, and we provide a teaching approach to encourage efficient diagrammatic reasoning. Finally, we evaluate teaching using the “effect of time axis.” As a result, the conventional knowledge of diagrammatic representations of probability was identified as “the rectangle represents the denominator in the probability calculation and the part bounded by the circle inside the rectangle represents the numerator.” Furthermore, it is suggested that the diagrammatic representation of conditional probability created based on conventional knowledge is effective for understanding conditional probability.

BACKGROUND AND RESEARCH PURPOSE

Modern society is highly complex and therefore contains a lot of uncertain information. As a result, probabilistic reasoning in decision-making is becoming increasingly important for citizens. Probabilistic reasoning refers to a technique of reasoning based on conditional probability, such as “the probability that event B occurs when event A occurs is 50%.” This inference is suitable for modeling events with many uncertainties, unlike logic reasoning that is deterministically derived as “it is B if A.” Therefore, the importance of teaching conditional probability is increasing in school mathematics.

Many students find the reduction of the sample space in the calculation process of conditional probability to be confusing. Ishibashi (2017) examined the difficulties faced by 80 Japanese second-year high school students in learning conditional probability. He found that 23.75% of these students were confused by the notations $P(B|A)$ and $P(A \cap B)$. Students set the denominator as $n(U)$ instead of $n(A)$ without paying attention to the condition that event A has occurred (which corresponds to the denominator of $P_A(B)$). In this context, reduction of the sample space means that the sample space is shrunk from whole event U to event A because it is already known that event A has occurred when obtaining the conditional probability of event B after event A has occurred. However, it is difficult to reduce the sample space from a linguistic expression of the conditional probability. It is important for students to understand this concept, and therefore, it is necessary to clarify the appropriate teaching approach for conditional probability.

Of course, some teaching approaches for conditional probability have been developed already. In fact, diagrammatic expressions such as Euler diagrams are used in high school mathematics textbooks in Japan (Takahashi et al., 2011). Larkin and Simon (1987) noted that diagrammatic representations are more effective than linguistic expressions because they mainly use spatial position indications in the context of information search and recognition. We think that they are also effective in modeling linguistic expressions of conditional probabilities when reducing the sample space. However, Ishibashi’s (2017) survey results clearly show that they are not fully used in currently used teaching approaches for conditional probability.

Therefore, in this study, we consider the use of diagrammatic representations as a teaching approach for conditional probability. We consider the “effect of time axis” that is often examined in research on the difficulties faced in teaching probability to students. The reason for choosing the “effect of time axis” is that when answering this question, the event after time becomes the condition for finding the probability of the previous event. However, modeling the linguistic expression into a probabilistic model is particularly difficult for learners, and this remains a difficulty for this problem (Fischbein and Schnarch, 1997).

EFFECT OF TIME AXIS

The problem of the “effect of time axis” is described as follows:

“Yoav and Galit each receive two white and two black marbles.

(A) Yoav extracts a marble from his box and sees that it is white. Without replacing the first marble, he extracts a second marble. Is the likelihood that this second marble is also white smaller than, equal to, or greater than the likelihood that it is a black marble?

(B) Galit extracts a marble from her box and puts it aside without looking at it. She then extracts a second marble and sees that it is white. Is the likelihood that the first marble she extracted is white smaller than, equal to, or greater than the likelihood that it is a black marble?” (Fischbein and Schnarch, 1997, p. 99).

The correct answer to this question is “small” (i.e., “smaller than”) for both problems A and B. A typical misrepresentation is to answer “small” for Problem A but “equal” for Problem B. In Problem B, it is often noted that “the second result will not affect the first result” and that because there are two white balls and two black balls initially, the answer given is “equal.”

The “effect of time axis” has been examined frequently in research on probability difficulties and probability understanding as a problem to which people intuitively give wrong answers. For example, Fischbein and Schnarch (1997) conducted a survey of the “effect of time axis” problem with fifth-, seventh-, ninth-, and eleventh-graders in Israel to evaluate their judgment of probability. We are investigating the actual situation. We found that the proportion who give wrong answers increases with school age, except for college students. Matsuura (2006) conducted a similar survey for fifth graders to university students in Japan and reported that only half of the students from any grade gave correct answers.

Fischbein and Schnarch (1997) believed that the difficulty faced with the “effect of time axis” arises from the tendency of human thought. They noted that such errors are “intuitively based misconceptions” and that it is assumed that later events will not affect previous events.

DIAGRAM

Semiotics based on Peirce’s epistemology is effective for considering the role of symbols such as diagrams (Hoffmann, 2005). Therefore, we first follow Peirce’s semiotics and outline the diagram and diagrammatic reasoning. Then, we critically consider the present curriculum of probability and conditional probability in Japanese school mathematics from the viewpoint of Peirce’s semiotics. We provide suggestions about diagrammatic expressions that encourage an understanding of the linguistic expressions of conditional probability.

Diagram and diagrammatic reasoning

Peirce defines a “diagram” as “a representation which is predominantly an icon of relations and is aided to be so by conventions. Indices are also more or less used. It should be carried out upon a perfectly consistent system of representation, founded upon a simple and easily intelligible basic idea” (CP 4.418, 1903; quotations from *Collected papers of Charles Sanders Peirce* have been expressed as (CP volume number, paragraph number)). Finding and understanding new properties and using them is called diagrammatic reasoning (Hoffmann, 2005), and it involves the “construction of a diagram” using familiar expressions, rewriting, thought experiments, “experiments,” and “observations” to recognize and understand new properties. This study aims to infer mathematical models from the linguistic expressions of conditional probabilities through diagrams.

Kosslyn (1989) cited the use of conventional knowledge as a component of diagrams for effective diagrammatic reasoning. Conventional knowledge refers to the rules for the abstraction of expressive objects that are inherent to each diagrammatic representation. For example, a matrix-type table contains detailed information about combinations of two item sets (Hurley and Novick, 2010). If it is necessary to express combinations of two types of item sets for problem solving, it is useful to express the problem using a matrix-type table in a diagram. On the other hand, if some information needs to be obtained from the matrix-type table, it is possible to efficiently show the

relationship between the constituent elements by considering what type of conventional knowledge the diagrammatic representation is based on.

A general diagrammatic representation shows individual components and their relationships. It is difficult to obtain accurate information from a diagrammatic representation without knowing any convention, and therefore, the conventional knowledge of how abstracting expressions are abstracted is the correct understanding of the diagrammatic representation. This is consistent with “simple and easily intelligible basic idea” in the definition of the Peirce diagram.

Conventional knowledge on probabilistic diagram

As mentioned above, to efficiently obtain information from a diagrammatic representation, it is necessary to be able to apply appropriate conventional knowledge for that diagrammatic representation. Thus, effective diagrammatic reasoning is performed. Based on the above, this section and the next section consider how conditional probability is currently taught in school mathematics. In this study, we hypothesize that the diagrammatic representation of conditional probability in textbooks does not work well, and therefore, it is important to understand the diagrammatic representation of conditional probability from a semiotics viewpoint.

In Japanese high school mathematics textbooks (mainly from primary school to high school students of age 15–16 years), Euler diagrams are used for diagrammatic representation of probabilities (Takahashi et al., 2011, p. 39 (Figure. 1), p. 44 (Figure. 2), p. 46 (Figures. 3 and 4), p. 48 (Figure. 5)).

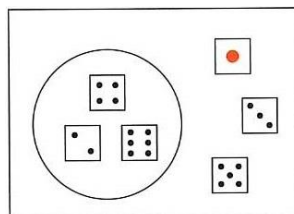


Figure. 1: Probability of throwing a single die to give an even number of rolls

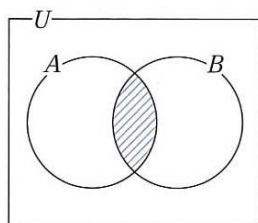


Figure. 2: Probability of product event

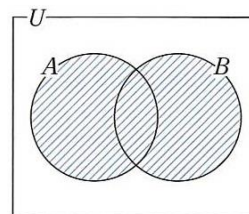


Figure. 3: Probability of sum events

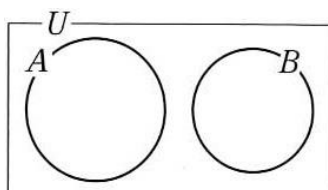


Figure. 4: Probability of exclusive events

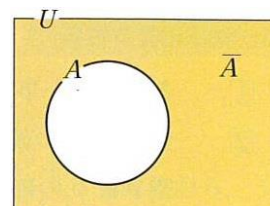


Figure. 5: Probability of complementary event

In school mathematics in Japan, Euler diagrams are first presented in “Set and Proposition.” We call the set of all things the “whole set,” denote it as “U,” and display it as a rectangle. Next, we consider the appropriate set in the whole set. For example, if the whole set contains all natural numbers, consider a set of multiples of 2, denote it as “A,” and display it as a circle. Hereafter, unless otherwise noted, it is assumed that the whole set is displayed as a rectangle and all sets displayed as circles show a subset of the whole set (Hashimoto, 2016). Hashimoto (2016) noted that Euler diagrams are considered a type of Venn diagram. Such diagrams provide students with a simple and easily intelligible representation for conventional knowledge.

Students can learn probability by using this conventional knowledge. Takahashi et al. (2011) introduced probability using Figure. 1:

$$P(A) = \frac{n(A)}{n(U)} = \frac{\text{Number of occurrences of event } A}{\text{Number of all possible cases}}$$

In Figure. 1, the rectangle represents the whole set for the throw of a single die, and the circle represents the event that an even number of rolls appears. This is consistent with the conventional knowledge mentioned above. Meanwhile, new conventional knowledge that was not in the set is added. That is, “the rectangle represents the denominator in the probability calculation, and the part bounded by the circle inside the rectangle represents the numerator.” Figures. 2–5 show some more diagrammatic expressions. If students use these repeatedly during learning for problems such as that shown in Figure. 1, they can acquire these diagrammatic expressions as conventional knowledge.

Critical consideration of diagrammatic representation of conditional probability

Figures. 1–5 show conventional diagrammatic representations. However, Takahashi et al. (2011) used the diagrammatic representation shown below for expressing the conditional probability (Figure. 6).

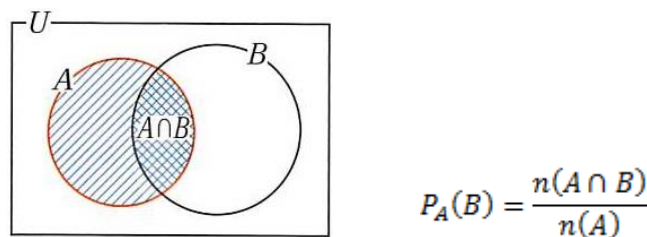


Figure. 6: Conditional probability (Takahashi et al., 2011, p. 58)

In Figure. 6, the denominator and numerator in the probability calculation are represented by the part bounded by the circle inside the rectangle. This is contrary to the definition of Peirce’s diagrammatic representation that it must be performed on a consistent representation scheme. Furthermore, considering that conventional knowledge is essential for understanding the diagrammatic representation (CP 4.418; Kosslyn, 1989), this unconventional diagrammatic representation (Figure. 6) may hinder students’ understanding of conditional probability. The above example suggests that the diagrammatic representation of conditional probability in textbooks is not one that sufficiently considers conventional knowledge, implying that the diagram used for representing conditional probability in the textbook does not work well. This proves the hypothesis of this study.

TEACHING CONDITIONAL PROBABILITY USING DIAGRAMS

Diagrammatic representation of conditional probability based on conventional knowledge

As described above, the diagrammatic representation of conditional probability in the textbook does not sufficiently consider the conventional knowledge of probabilistic diagrams. Therefore, in this section, we develop a diagram of conditional probabilities based on conventional knowledge and verify its validity through class practice.

Based on conventional knowledge, the conditional probability diagram can be constructed as follows (Figure. 7).

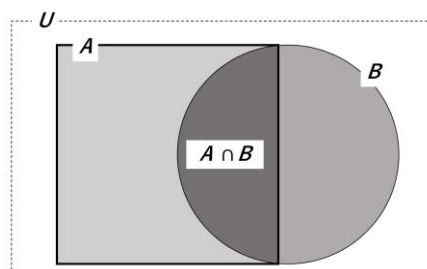


Figure. 7: Diagrammatic representation of conditional probability based on conventional knowledge

In Figure. 7, a rectangle represents the denominator ($n(A)$) in a probability calculation, and a circle represents the numerator ($n(A \cap B)$). In Figure. 6, it is represented as a rectangle for whole event U, which is a diagrammatic representation of the denominator based on conventional knowledge. Because whole event U was found to be the subject of consideration as event A was found to occur, this was shown in Figure. 7 by a dotted line.

Teaching for use of diagrammatic reasoning

To clarify the role of Figure. 7 (as an external representation) in learning processes, it is effective to follow diagrammatic reasoning, by which one can explain the development of knowledge on the basis of a three-stage activity: constructing representations, experimenting with them, and observing the results (Hoffmann, 2005). If three stages of diagrammatic reasoning are intended, teaching can be designed with the following flow (Figure. 8).

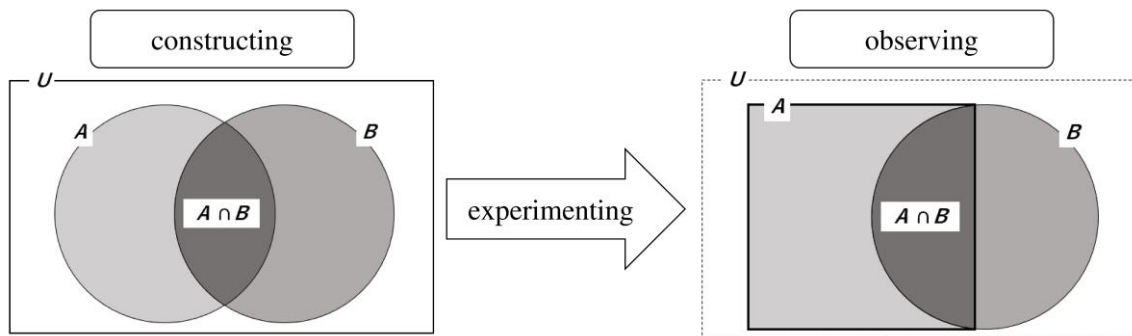


Figure. 8: Process of diagrammatic reasoning based on example shown in Figure. 7

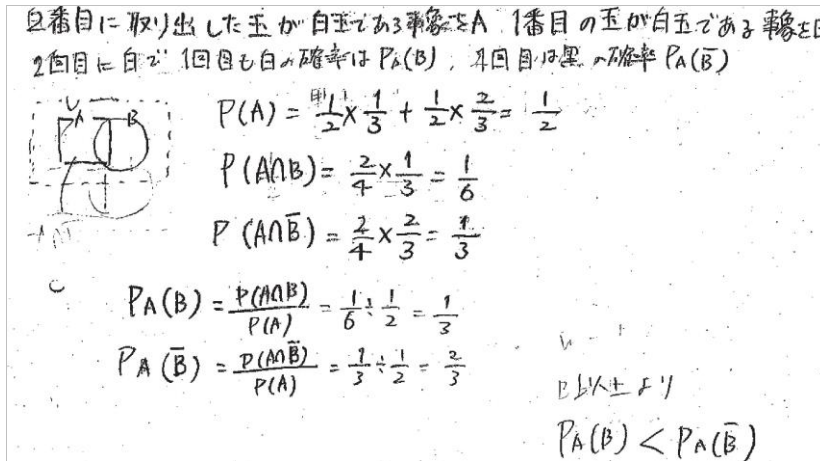
First, because familiar expressions in the composition stage of the diagram refer to expressions based on conventional knowledge in this paper, we use Euler diagrams to represent the product phenomenon (Figure. 2). Here, the sample space refers to event U. In the second experimental stage, given the condition that, for example, event A occurs, we manipulate the diagram to reduce the sample space. At this time, Figure. 2 is drawn and expressed as shown in Figure. 7. This emphasizes that the sample space has been reduced owing to the information that has been obtained. After the operation as an experiment, in the third observation stage, based on conventional knowledge, an understanding of conditional probability is encouraged by using the rewritten diagram.

Survey summary and results

The survey was conducted with 39 first-grade high school students (age 15–16 years) who learned conditional probability by using the teaching approach shown in Figure 8. The survey problem was the “effect of time axis.”

In this class, first, we shared the conventional knowledge of the diagrammatic representation of probability (the rectangle represents the denominator in the probability calculation and the part bounded by the circle inside the rectangle represents the numerator). Next, we presented the following conditional probability problem from the textbook: “Take 1 ball out of a box containing 9 balls of which five balls are numbered 1, 2, 3, 4, and 5 and four balls are numbered 6, 7, 8, and 9. What is the probability of getting an odd-numbered ball given the removed ball is red?” Then, we let the students think about the difference from the probability problems they have been thinking about thus far, and we demonstrated how the sample space is reduced based on the process shown in Figure 8.

The survey result showed that the ratio of the number of people who correctly answered both problems A and B was 46%. By contrast, Fischbein and Schnarch (1997) and Matsuura (2006) conducted surveys and reported correct answer rates of 30% (with 20 people) and 36% (with 38 people), respectively. Because Fischbein and Schnarch (1997) did not cover the tenth grade, which is the first grade in Japanese high schools, it is compared with the survey result for the 11th grade here. As shown in Figure. 9, in addition to students’ intuition, modeling and processing using diagrammatic reasoning helped students process the “effect of time axis,” resulting in a high correct answer rate.



Event A is “The second marble is a white marble.”

Event B is “The first marble is a white marble.”

The probability that the first marble is a white marble when the second marble is a white marble is $P_A(B)$.

The probability that the first marble is a black marble when the second marble is a white marble is $P_A(\bar{B})$.

Figure. 9: A description given by a student who solved the “effect of time axis” using a diagram (The English sentence on the right in Figure. 9 is an English translation of the student's description.)

CONCLUSION

Under the hypothesis that the diagrammatic representation of conditional probability in the textbook is not sufficient, this study uses Peirce’s semiotics definition for diagrams to consider a diagrammatic expression that is effective for understanding conditional probability. As a result, first, because the diagrammatic representation in textbooks is not based on conventional knowledge of probabilistic diagrammatic formulation, it was revealed that it is insufficient for understanding conditional probability. Second, we developed a diagrammatic representation of conditional probability based on conventional knowledge and showed that students could use it effectively for understanding conditional probability by considering the “effect of time axis” problem as an example.

This paper focused on conditional probability. In the future, I would like to develop a systematic approach for teaching probability through diagrammatic representations.

REFERENCES

Fischbein, E., and Schnarch, D. (1997). The Evolution with Age of Probabilistic, Intuitively Based Misconceptions, *Journal for Research in Mathematics Education*, 28, 97-105.

Hashimoto, K. (2016). Set Operations and Venn Diagrams. *Studies in Philosophy*, 41, 1-26.

Hoffmann, M.H.G. (2005). Signs as Means for Discoveries. Peirce and His Concepts of “Diagrammatic Reasoning,” “Theorematic Deduction,” “Hypostatic Abstraction,” and “Theoric Transformation.” In Hoffmann, M.H.G., Lenhard, J., and Seeger, F. (Eds.), *Activity and Sign: Grounding Mathematics Education* (45-46). Springer.

Hurley, S.M., and Novick, L.R. (2010). Solving Problems Using Matrix, Network, and Hierarchy Diagrams: The Consequences of Violating Construction Conventions. *The Quarterly Journal of Experimental Psychology*, 63, 275-290.

Ishibashi, I. (2017). A Study of Teaching Probability to Train Competency of Judgement Using Probability in Making Decision: Focusing on Bayes Theorem. *Journal of JASME Research in Mathematics Education*, 23 (2), 83-90.

Kosslyn, S.M. (1989). Understanding Charts and Graphs. *Applied Cognitive Psychology*, 3, 185-225.

Larkin, J., and Simon, H. (1987). Why a Diagram is (Sometimes) Worth 10,000 Words. *Cognitive Science*, 11, 65-99.

Matsuura, T. (2006). The Longitudinal and Cross-Sectional Study on the Probability Judgment in Children. *Journal of JASME Research in Mathematics Education*, 12, 141-151.

Peirce, C.S. (1931-35, 1958). *Collected Papers of Charles Sanders Peirce*. Harvard University Press.

Takahashi, Y., et al. (2010). *Mathematics A*. Keirinkan.