REASONING ABOUT VARIATION OF A UNIVARIATE DISTRIBUTION: A STUDY WITH SECONDARY MATHEMATICS TEACHERS

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Variation is a fundamental concept in statistics literacy; standard deviation is part of compulsory school curriculum in Brazil. The objective of this study is to explore reasoning about variability by teachers, using the model proposed by Garfield (2002). The sample was composed of nine in-service mathematics teachers who took part in a teacher-training course on statistics. An experimental focus made it possible for them to experience all the steps of a statistics research project in which the course content was designed to expose the reasoning about variability employed by these teachers. We identified an oscillation between idiosyncratic and procedural levels, but no teacher showed complete reasoning about variation. The most prevalent reasoning employed was verbal, when teachers interpreted standard deviation as a measure of variation among observations.

INTRODUCTION

The importance of the role of variation in statistics has been documented by a variety of researchers (Moore, 1997; Snee, 1990; Watson & Kelly, 2002) and is considered by Wild and Pfannkuch (1999) as the heart of their model of statistical thinking. This is why it is important to understand how people reason about variability. Hence, the present study aimed to explore the reasoning about variation of in-service mathematics teachers when exploring a univariate distribution. Although some research studies distinguish between variation and variability, in this work we define reasoning about variation as the cognitive processes involved in describing both the propensity for change and measure of variation.

Some research has already presented teachers' reasoning about variation, such as Canada (2006) in a probability context, Hammerman and Rubin (2004) in binning and proportional strategies in the analysis of data using a computer tool, and Makar and Confrey (2005) with prospective teachers to document the different types of language used to express notions of variation. Other research studies have been conducted on data and graphs with students such as Ben-Zvi (2002, 2004), Lehrer and Schauble (2002) and Reading (2004) and in other contexts (sampling, probability, data and graphs) such as Meletiou (2000), Watson and Kelly (2002), Watson, Kelly, Callingham and Shaughnessy (2003) and Reid and Reading (2006).

Some reasoning about variation studies have developed a hierarchy of consideration of variation from students' responses (Reading, 2004; Reading & Shaughnessy, 2004; Reid & Reading, 2006; Torok, 2000; Watson et al., 2003). In this study, we adapted to reasoning about variation the model of statistical reasoning by Garfield (2002) that consists of five levels (idiosyncratic, verbal, transitional, procedural and integrated process reasoning) so that we could identify common misconceptions regarding the measure of variation and the difficulties that teachers encounter when dealing with variability in data.

METHOD

The research sample was composed of nine mathematics teachers taking part in a teacher-training course. Their ages ranged from 26 to 59 years; two of them were male and seven female; eight had degrees in mathematics and one in chemistry, and all were teaching mathematics in middle and high school. Six of these teachers claimed to be teaching statistical content in their lessons. One of them also specifically stated that he taught arithmetic mean and standard deviation.

The training course was 48 hours in duration and was divided into 16 three-hour meetings. All of the sessions were tape recorded and transcribed. We worked using the contents

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of simple frequency distributions and grouped data, different graph representations, central tendency and dispersion measures, and Chebyshev's inequality, but the main subject of the course was variation.

This work followed the plan of an action research defined by Tripp (2005, p.447) as an investigation method that uses the techniques from solid research studies to inform the action taken to improve practice. The researcher conducting the course, referred to here as PSQ, played the role of mediator, encouraging the teachers to get involved, to participate, to commit themselves and to produce knowledge. We had the collaboration of some observers, referred to in this paper as OBS. Together, the teachers organized an empirical study, and the data obtained were used to develop the statistical content of the course. In this study, we present the analysis of the reasoning about variation when teachers were analysing the age of the people they had interviewed (108 completed questionnaires).

In this paper, the teachers are identified by their initials and as teachers. The respondents to the questionnaire are referred to as people. The teachers designed the frequency distribution for the ages (Table 1) and the associated histogram. They were then asked to think of a different way to represent the age. Immediately, they came up with the idea of calculating the arithmetic mean, which resulted in the value 38.6 years old.

Table 1. Frequency distribution with the grouped data of the variable age

People age	Frequency
19- 29	21
29 -34	10
34- 39	22
39 -44	22
44 -49	14
49- 54	11
54 -60	8
Total	108

Reasoning about variation was explored at three different times: when teachers were analysing the mean; when they perceived the need for a measure of variation; and when they tried to grasp the meaning of the standard deviation, a measure of variation chosen and calculated by them (9.8 years old).

According to Reading (2004, p. 85), "the study of variation in schools, such as standard deviation, is notorious among teachers as being particularly cumbersome, resulting in many teachers having difficulty in teaching the concept to students or avoiding it altogether." One of the explanations for this is that the understanding of standard deviation encompasses many statistical concepts such as arithmetic mean, deviations from the mean, and relative density of the data around the mean, as reported by Delmas and Liu (2005); these concepts are considered in this work about variation as an integrated process reasoning.

IDIOSYNCRATIC REASONING

According to Garfield (2002), reasoning is idiosyncratic when the student knows some statistical terms and symbols but uses them without complete understanding and, frequently, incorrectly. This kind of reasoning was observed when the teachers were asked to interpret the mean age of their interviewees. One of teachers, SB, asserted, "…that I would get to the school and the teachers would have this characteristic."

Interpreting the mean as the value that represents the age of the people who took part in the research is partially correct, but this value should be associated with a measure of variation. This kind of reasoning is analogous to the first level of Reid and Reading's hierarchy, where students do not consider variation.

When teachers realised they needed to use a measure to complement the arithmetic mean, teacher OB said, “I thought it was only the mean, you know?... You see, we still can calculate the standard deviation because it is still too spread out like this.” Unlike the reasoning of teacher SB, during the calculation of the mean teacher OB observed that the ages were very different from each other, and she indicated the need to calculate the standard deviation. The perception of variation in data was considered verbal reasoning, but the use of the term standard deviation was considered idiosyncratic because the teacher had already conducted a search through high school mathematics text books and realized that standard deviation existed but had not grasped its meaning, as shown by the excerpt below.

> OB: Look, PSQ! I told him [referring to the dialogue with teacher AM], then, we need to see what the dispersion is, the measure of dispersion. Then, I read in the book mean deviation, variance and standard deviation. Then, I told him that the standard deviation is the one that gives the most correct idea of dispersion. But, what now? If the mean is, in this case, I think it is 49, I have a mean deviation of 1.8, variance of 4.3 and standard deviation of 2. I don’t understand anything [referring to example in the book].

Another factor deserves underscoring. The expression ‘it is too spread’ out used by the teacher OB referred to variation as predicative, and, according to Makar and Confrey (2005), this concept is only grasped when it is interpreted as a noun, such as the variation is big.

When PSQ was trying to emphasise that the standard deviation is a measure of variation around the mean, teacher AM explained that standard deviation of zero means that all data are the same (considered verbal reasoning), but this fact emerged through a misunderstanding between standard deviation and mean equals zero, as can be observed in teacher LH’s question, “What is zero, the mean or the standard deviation?”

VERBAL REASONING

According to Garfield (2002), reasoning is verbal when there is an understanding of certain concepts, but one cannot apply them to actual behaviour. In this work, four acts of verbal reasoning were observed: perception of variation; understanding the standard deviation as the measure of the difference between the values of the data; the idea that low standard deviation is better and recognizing there are quantities of values within one standard deviation interval from the mean.

When teachers were trying to understand the mean, teacher OB said, “If the mean is 39 years old, most of them are 39 years old, but not all of them are 39. Now, as the mean is 39, we know that there are a few 19-year-olds, and a few 60-year-olds, 40, 43, 35; you can more or less get the idea.” Although this teacher considered the possibility of the existence of some variation, she grasped the mean as mode, a misunderstanding of mean described by Batanero (2000) and estimated the format of distribution using the mean and extreme values.

The initial understanding of standard deviation was related to the measure of the difference between the values of the data, as the following dialogues show.

> AM: Standard deviation will point to, will show, will be an alarm, will say: oops! There is a dispersion of points here. It shows the irregularity.
> IS: And with the variation of an age to another of approximately 9 years old.
> AM: Next to zero the data are more concentrated.

The interpretation of the standard deviation as homogeneity of the sample makes it difficult to understand this measure of variation around the mean. However, high school mathematics textbooks emphasise this conception (Loosen, Lioen & Lacante, 1985). This kind of reasoning is poorer than the third level of hierarchy of Reading and Shaughnessy (2004), where the students discussed deviations from an anchor not necessarily central.

Another identified form of verbal reasoning was the preference for a low standard deviation. The teachers said that the value for the standard deviation helps to interpret the results, but it was not evident that they understood that the values of the sample tend to be closer to the mean as standard deviation moves toward zero. An example can be seen when AM affirms, “Calculate the standard deviation, and the low value means that everything is all right. But if the standard deviation is too high, it shows that the ages are too spread out, so then I have to worry about these ages here.”

As the teachers make progress in their reading, they realise that the books address intervals around the mean, and, for that, they used standard deviation. To grasp this interval, the teacher RN made an analogy with poll and said, “Well, I think that we should work with percentages, something percent would be there... Or this many people, even from 108, this many people....” It is possible to observe that she intuitively supposed that there were some quantities of people within it, but she did not know how many.

TRANSITIONAL REASONING

According to Garfield (2002), transitional reasoning is the ability to correctly identify one or two dimensions of a statistical concept, without completely integrating them. In this sense, in the cases in which the teachers used more than one summary measure to understand the variation of the distribution adequately, we considered this to be transitional reasoning. It is similar to a second level of description of variation used by Reading and Shaughnessy (2004). The teachers used maximum and minimum values, mode and the graph representation itself, confirming the results obtained by Ben-Zvi (2004) and Lehrer and Schauble (2002), in regard to the lack of need for a measure of variation.

The first suggestion presented by teachers OB, AM and LH was a simple presentation of a histogram, along with the mean, leaving the task of analysing the variation to the reader. Teacher LH said, “You get a strip down, a strip up. See the number of people, the age it starts, and I don’t know.” ‘Calculate three strips, and calculate’; one would think that he was referring to the interval of one standard deviation of the mean. However, during the discourse, it became apparent that he was referring to the bars of the histogram of the frequency distribution presented in Table 1. It is important to note that he was very enthusiastic about the research he had done on the subject.

Based on the modal group of the frequency distribution, teacher RS wrote the following phrase, “After conducting statistical studies (...) we have reached the conclusion that the ages of the people range from 19 to 60 years old, and most of them were between 34 and 44 years old.” This phrase was categorized as transitional reasoning because it used the modal group and the maximum and minimum values of the distribution. It is important to emphasise that the strategy of using these values had been used before in other activities in this research and also in studies by Reading (2004), Meletiou (2000) and Ben-Zvi (2004).

It is interesting to observe that this reasoning is completely independent from raising the need to use mean and standard deviation, concepts already discussed in the group. When PSQ presented Chebishev’s inequality, it was observed that about 95% of the interviewees were between 18.9 and 58.1 years old and the following debate arose:

> RN: How do we consider this fact that the most of the people are between 34 and 44?
> OBS1: When you calculate the standard deviation, one up and one down, you have the majority. And it is all the same...
> PSQ: You’ve already got the majority.
> OBS1: You said from 34 to 44, didn’t you? It is like this: what are you interpreting as a majority? 50% plus 1? So, then, I think you wouldn’t need to do the entire statistical study. You would raise all the ages.

Teacher RN was referring to the majority obtained from the observation that the two groups present higher frequencies (the group from 34 to 39 years old has 22 people, and the group from 39 to 44 also has 22 people), representing 44 of the 108 people in the study, which means approximately 41% of the sample. When PSQ was talking about majority, she was reporting the majority in the interval of two standard deviations from the mean (95%), and the teacher OB was talking about the majority using the range, which represents the interval containing 100% of the data.

Confusing the mean with mode (majority) and the use of a modal group of a distribution make reasoning about variation increasingly difficult.

PROCEDURAL REASONING

According to Garfield (2002), procedural reasoning is the ability to correctly identify the dimensions of a statistical concept or process, without completely integrating them or without understanding the process. When the teacher grasped the meaning of mean, deviations from this mean and started to design an interval from the mean, we considered this to be procedural reasoning. This reasoning was not considered to be complete, because the teachers did not understand that they could estimate a percentage of the observations in this interval.

> PSQ: Suppose that half of the people were 20 years old and half were 60 years old; would the standard deviation be close to zero or far from it?
> SB: Nobody would be 40 years old. We would have a mean of 40 years old and no one would be 40.
> PSQ: What happens with each age? What happens when you have more data far from the mean?
> OB: More dispersion!
> PSQ: So, what happens to the standard deviation?
> OB: It is going to be high!
> PSQ: So, let's go back to our exercise. We are not going to tell an ET that the mean age is 39. If it has a standard deviation of 1 year, so...
> OB: So they are 37, 40.
> PSQ: If I say that these people have a mean age of 39 years old, with a variation of 30, a standard deviation of 30, then there are people...
> OB: Very Young and very old.

From this discussion, teachers AM and CI formulated the following phrase to interpret the age of their research participants:

> A study (...) has shown that the mean age is approximately 39 years old. The modes, meaning the ages that most frequently appear are 39 and 41 years old (7 times each) and are close to the mean. The standard deviation is calculated to be the value 9.8, which means that on the mean there is a variation of 10 years above and 10 years below, that is, the range in variation is from 29 to 49 years old.

It is possible to observe that the teachers began to elaborate the variation interval around the mean, but they were not clear about what it meant.

FINAL CONSIDERATIONS

None of the teachers integrated process reasoning, which would relate the understanding of mean, deviations from the mean, the interval of *k* standard deviations from the mean and the density estimation of frequency in that interval. The results obtained with these teachers are similar to those published on student reasoning about variation, as reported by Hammerman & Rubin (2004), indicating the need to rethink the initial and continuing teacher training courses for mathematics teachers.

The predominant reasoning about variation was verbal, with the understanding that standard deviation is a measure of sample homogeneity. Being limited to the verbal reasoning about variation does not allow mathematics teachers to teach their students the meaning of measures such as standard deviation, restricting them to the teaching of algorithms. This was emphasized by teacher AM, when he said that he only taught how to calculate the mean, median and the standard deviation, but he had never thought about how these concepts could be related.

REFERENCES

Batanero, C. (2000). Significado y comprensión de las medidas de posición central (Meaning and understanding of central tendency measures). *UNO, 25*, 41-58.

Ben-Zvi, D. (2002). Seventh grade students' sense making of data and data representations. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics*. [CD-ROM]. Voorburg, Netherlands: International Statistical Institute.

Ben-Zvi, D. (2004). Reasoning about variability in comparing distributions. *Statistics Education Research Journal*, *3*(2), 42-63. Online: www.stat.auckland.ac.nz/~iase/serj/.

Canada, D. (2006). Elementary pre-service teachers´conceptions of variation in a probability context. *Statistics Education Research Journal*, *5*(1), 36-63. Online: www.stat.auckland.ac.nz/~iase/serj/.

DelMas, R., & Liu, Y. (2005). Exploring student's conceptions of the standard deviation. *Statistics Education Research Journal*, *4*(1), 55-82. Online: www.stat.auckland.ac.nz/~iase/serj/.

Garfield, J. (2002). The challenge of developing statistical reasoning. *Journal of Statistics Education*, *10*(3), 1-11. Online: amstat.org/publications/jse/.

Hammerman, J. K., & Rubin, A. (2004). Strategies for managing statistical complexity with new software tools. *Statistics Education Research Journal*, *3*(2), 17-41. Online: www.stat.auckland.ac.nz/~iase/serj/.

Lehrer, R., & Schauble L. (2002). Distribution: A resource for understanding error and natural variation. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics*. [CD-ROM]. Voorburg, Netherlands: International Statistical Institute.

Loosen, F., Lioen, M., & Lacante, M. (1985). The standard deviation: Some drawbacks of an intuitive approach. *Teaching Statistics*, *7*(1), 2-5, 1985.

Makar, K., & Confrey, J. (2005). "Variation-talk": Articulating meaning in statistics. *Statistics Education Research Journal*, *4*(1), 27-54. Online: www.stat.auckland.ac.nz/~iase/serj/.

Meletiou, M. (2000). *Developing student's conceptions of variation: An untapped well in statistical reasoning*. Unpublished Ph.D. University of Texas, Austin.

Moore, D. (1997). New pedagogy and new content: The case of statistics. *International Statistical Review*, *65*(2), 123-137.

Torok, R. (2000). Chance & data. *Australian Mathematics Teacher*, *56*(2), 25-31.

Tripp, D. (2005). Pesquisa-ação: Uma introdução metodológica (Action research: A methodological introduction). *Educação e Pesquisa*, *31*(3), 443-466.

Reid, J., & Reading, C. (2006). A hierarchy of tertiary students´consideration of variation. In Allan Rossman & Beth Chance (Eds.). *Proceedings of the Seven International Conference on Teaching Statistics*. [CD-ROM]. Voorburg, Netherlands: International Statistical Institute.

Reading, C. (2004). Student description of variation while working with weather data. *Statistics Education Research Journal*, *3*(2), 84-105. Online: www.stat.auckland.ac.nz/~iase/serj/.

Reading, C., & Shaughnessy, J. M. (2004). Reasoning about variation. In D. Ben-Zvi & J. Garfield (Eds), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 201-226). The Netherlands: Kluwer Academic Publishers.

Snee, R. (1990). Statistical thinking and its contribution to total quality. *The American Statistician*, *44*(2), 116-121.

Watson, J., & Kelly, B. (2002). Can grade 3 students learn about variation? In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics*. [CD-ROM]. Voorburg, Netherlands: International Statistical Institute.

Watson, J., Kelly, B., Callingham, R., & Shaughnessy, M. (2003). The measurement of school students´ understanding of statistical variation. *International Journal of Mathematical Education in Science and Technology, 34*(1), 1-29.

Wild, C., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, *67*(3), 223-265.