Teaching probability at secondary schools using computers

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1 Introduction

The use of computers in teaching at secondary schools is quite often discussed nowadays. Most of the schools are endowed with modern computer laboratories and ask for learning software, tutorials, methodical books, etc. It is evident that these materials should be designed to increase the efficiency of the educational process and not to lead to a mere brainless playing with computers only. Therefore, our principal goal was to combine teaching probability with elements of algorithmization and computer programming. The idea is obviously not new. There exist some successful projects concerning computer aided teaching probability theory and statistics in university courses, see, e.g., list of projects mentioned in references. However, only a small number of projects deals with teaching probability at secondary schools.

We address three different types of audience:

1. *Teachers of conventional classes*, who want to variegate their lessons. The students of today like working with computers and enjoy using of nontraditional and modern teaching methods. If properly supervised and guided, they can understand important basic principles of the probability through the trial and error method and, at the same moment, try to guess the results of given problems and verify them by computer simulation. The next step is, evidently, to discover basic probability principles and phenomena. For this purpose probability theory provides enough attractive problems. In addition to its practical importance the probability is, without any doubt, one of the most entertaining branches of mathematics. The general idea behind our project is thus the hope that students will find probability both interesting and useful and, perhaps, they will choose stochastics for their future study.
2. *Teachers of mathematical classes*, leaders of mathematical bees, camps and solving seminars. More generally all those who prepare students for mathematical competitions. There is a significant difference between what is taught at most of the secondary schools through standard lessons of mathematics and what is expected from the participants of mathematical competitions. For them we included more advanced notions and topics as *random variables* and *random walks*. All these problems have been collected by the authors during the long standing work with mathematically talented students.
3. *Mathematically talented students*, who cannot attend a mathematical bee or solving seminar at their secondary school. For them we prepared a collections of tasks related to some interesting problems. At first, these tasks are arranged to lead students to discover basic probability principals and/or phenomena. At second, multiple solutions to problems are provided in order to give students possibility to compare different problem solving strategies. At third, many exercises

are available that enable students to practice solving methods and strategies and understand basic principals in more details.

The amount of the study material is rather great. It is a pity that there is not enough time for probability, and stochastics as well, in contemporary secondary schools curricula, at least in our country. From this point of view we do not expect that the teacher familiarizes students with all the material. Wide variety of topics enable the teacher to choose content of his/her lessons according to his/her own pedagogical attitude and experience.

2 Teaching probability (at secondary schools)

From the methodical point of view we accepted the theory of cognitive process, which can be summarized in the following four stages:

1. Motivation.
2. Acquiring experience.
3. Birth of knowledge.
4. Crystallization and automatization of knowledge.

In our opinion these four stages should be respected when teaching not only at secondary schools.

The motivation stage is necessary initiate for the cognitive process. It is clear that careful choice of probability problems to be solved is very important for this purpose. These problems should be attractive, meaningful and suitable for the students. The natural process of acquiring experience is one of the most important concepts in learning mathematics at the secondary school level. Especially in the field of probability students lack practice with phenomena of the chance. Unfortunately, without understanding this phenomena students do not understand the theory and their knowledge is formal. These problems are usually realized when students meet a paradox, untypical task or if they must apply the theory in real life.

Experiments such as tossing a coin, rolling a dice, spinning a roulette wheel, taking a card from a well-shuffled deck are suitable for acquiring experience with the phenomena of the chance. Here is an example: In the first lesson of probability students are equipped with coins, dices and a deck of poker cards. The teacher opens the lesson with three introductory problems:

a. What sum of the points is most often obtained if two dice are rolled repeatedly?
b. What is the chance of obtaining three heads if three coins are tossed?
c. What are the odds that two cards of the some denomination appear in five-card poker hand taken from a well-shuffled deck?

Students work in pairs. The first performs trials repeatedly and the other records the results. After performing a certain number of experiments, students formulate hypotheses and compare them with other students. Practical realization of such experiments is highly time-consuming. However, with the aid of computers, thousands of such experiments can be simulated very fast. Therefore, computer simulations are often used instead.

The moment of a birth of knowledge has an important role in the process of forming human psyche. This is the moment of strong feeling of a joy. The teacher should not miss this unique opportunity to encourage students' interest in mathematics. However, without previous two stages this moment cannot be reached. For this reason the problems must be sorted in order to give students sufficient experience and then lead them to discover new pieces of knowledge.

The last stage of the cognitive process involves crystallization and automatization of the knowledge. Therefore, it is important to provide students with many exercises to evolve their knowledge and skills.

3 Teaching computer programming (at secondary schools)

Most of the projects related to the computer aided probability and statistics teaching present simulation algorithms only as an instrument for demonstration of the chance phenomenon, i.e., students are not forced to create their algorithms. We decided to use a different approach which resulted from our experience in supervising computer programming courses for beginners. It suggests that creating and/or modifying simulation algorithms is of great mean to familiarize students with the essential foundations of programming.

In any case, natural question which programming language is suitable for this purpose arises. On one hand it had to be designed for mathematical calculations and to have graphical tools for visualization of the results. On the other hand this language should be simple to learn for secondary school students. There are several powerful mathematical systems available such as Derive, Maple, Mathematica, Matlab, etc. However, they are rather complicated and, in addition, they are very expensive, at least for our secondary schools.

While looking for a cheap but high-quality system, we realized that the computer algebra system MuPAD was available for free for educational institutions. This system has a simple programming language similar to Pascal, many built-in mathematical functions and a congenial graphical tool. Its help system is also very sophisticated. It consists of *dvi* file viewer with hypertext functionality and direct linking to MuPAD. This enables the users of TEX to create high-quality interactive teaching materials. Aside that, when studying the electronic interactive text, the reader can run all included algorithms simply just by a mouse-click.

4 Basic probability concepts covered

In this section we specify most important probability and programming concepts used in our approach together with some remarks and comments.

Notice that the conversion of our codes into the MAple codes is very easy thanks to the resemblance between both languages. As pointed out by Z. Karian, who had available longer version of this paper, the transcription was immediate. Analogously we can imagine conversion of our codes into the language of Mathematica or Matlab, e.g. However. the inner logic of these languages seems to us quite complicated, especially for beginners in programming of basics mathematics who prevail in secondary schools.

4.1 Introduction to MuPAD

The main goal is to introduce the computer algebra system MuPAD and to show how to control MuPAD environment efficiently. For this purpose some examples of numerical and symbolical computations have been prepared, such as counting all digits of 2^{1234}, looking for the prime factors of the huge numbers or factoring algebraic expressions. Demonstration of MuPAD 2D and 3D graphics is also included together with description of the MuPAD Help Tool. General attitude is to show the students different possibilities of the MuPAD system that can help them to understand better some other parts of mathematics taught at the secondary school.

4.2 Computer simulation methods

This is an introduction to the computer simulation methods. Students learn how to use random number generator to simulate tossing a coin, rolling a dice, spinning a roulette wheel or taking a card from a well-shuffled deck. They familiarize basic MuPAD programming structures such as `for,` `while` and `repeat` loops and `if-then-else` and `switch-case` branching instructions. With a view to saving data from simulation experiments students learn how to handle variables and arrays. Finally, they make the acquaintance with the visualization of their results using the MuPAD graphics tool. Many problems, which are designed to demonstrate all the above mentioned elements of the MuPAD programming language, are included here.

4.3 Probability

The aim of the previous part was to offer the students an occasion to acquire enough experience with phenomena of the chance. While solving the problems by computer simulations, students understand several basic principles concerning relative frequency of outcomes of random experiments. After this preliminary stage students are prepared to learn relevant topics from the probability theory.

Basic concepts of probability as outcomes, events and sample space of random experiment are defined as is usual in the textbooks for the secondary school level. However, more emphasis is given on making interconnections between new ideas and students' previous knowledge of elementary set theory. For this reason all concepts such as inclusion, intersection, union and difference of events are illustrated on several types of Venn diagrams. Exercises to practical construction of a sample space and events related to the experiment are also included.

The most important is definition of probability, which is introduced by the reference to the previous knowledge. Relative frequency motivation is also used for the definition of probability of an event. Classical definition of the probability results as an evident consequence.

4.4 Properties of probability

The main goal is to cover several important prepositions concerning the probability. In the effort of respecting theory of cognitive process described in Section 2, special care is devoted to the selection of motivation problems. Before announcing a new preposition or theorem, one or more examples are solved using computer simulation method. According to our experience acquired from real teaching, students usually discover the idea of preposition before teacher "announces it". The moment of discovering some principle or idea is very stimulating for students as mentioned in Section 2. The most important theorem of this part concerns addition of probabilities.

4.5 Independent trials

Concept of independent trials can be understood as a special case of more general concept of independent events. Therefore, in advanced textbooks it is usually skipped or just briefly mentioned within the topic of conditional probability and independence. Unfortunately, in the secondary school curricula this topic comes as optional and, due to the limited time space, it is often omitted. However, without introducing the notion of independent trials many elementary problems must be solved in rather complicated way. For example, important concept of Bernoulli scheme that has many practical applications is based on ideas of independent trials and multiplication of probabilities. For this pedagogical reason we devote separate space to this topic which should precede optional and more general notions of conditional probability and independence.

4.6 Conditional probability and independence

In this part we start with several simple programming exercises that help students to understand the difference between particular trials and compound trial. Another aim of these exercises is to show students some useful programming techniques. For many of them it is rather surprising that one of the presented algorithms can be used, after slight modification, for conversion of numbers from decimal to binary system, etc.

While topics presented in the previous parts are usually included in compulsory part of secondary school curricula, the concept of the conditional probability belongs to the optional part. Therefore, thanks to the permanent lack of time, teachers often omit it. In authors' opinion it is a real pity because the understanding of relationship between conditional probability and independence enable students to gather more general view about probability. According to our point of view, conditional probability can be taught in two different ways. The first way is shorter. The teacher briefly introduces two main concepts, i.e. the conditional probability and independence and then typical secondary school problems are solved. Usually both computer simulation and theoretical calculation solutions are presented and their results compared. More advanced topics such as the Bayes theorem and independence of three or more events are skipped. This amount of subject matter can be captured within common classes. The second way is intended for mathematical classes or mathematical seminars. After introduction of conditional probability, the *theorem of the total probability* and the *Bayes theorem* are formulated and proved. Then several interesting and practical problems, for example from medical research, are solved.

4.7 Random variables

This optional part starts with the definition of discrete random variable and distribution of probability. These two important concepts are illustrated by many examples. After that, basic ideas of binomial and geometric distribution are introduced using computer simulations. Results of these simulations are always plotted using the bar charts which help students to create visual conception of individual distributions. Continuous random variables are not discussed. After this introductory part the students are ready to make acquaintance with all important ideas of the expected value of a random variable. It is again motivated by computer simulations.

4.8 Random walks

In this optional part relatively untraditional method of solving some kinds of probability problems is presented. It has been created by virtue of very inspirative article of Engel (1975). The basic idea is that many probability problems can be visually modeled by random walks on graphs. Thanks to this "embedding" many rather difficult problems can be solved more easily. Students learn how to calculate the probability of the absorption representing typically the end of the game, as well as the mean time to this absorption among others.

5 Verification of the teaching method in practice

Methodical materials have been verified repeatedly during the second author's pedagogical practice at the Gymázium (grammar school) of F.M. Pelzl in Rychnov nad Kněžnou. Experimental teaching was performed with four different groups of students.

- *Conventional class.* There are typically thirty students in this class from which roughly one third chose mathematics for the school-leaving exam. During last year of studies, when probability is usually taught, the following teaching model was used: In three consecutive weeks students spent one lesson performing simulation experiments in a computer laboratory and then two lessons learning theory and solving problems using conventional methods. From the theory only compulsory topics were discussed, while conditional probability, random variables and random walks were omitted. Programming was not taught in these classes because of the lack of time. Because of small number of workstations (16 computers) in the computer laboratory, this class was divided into two groups. The first group worked at the time of regular lessons, whereas the second group worked in the morning before regular lessons. Relatively a small number of students in groups gave the teacher the possibility to monitor work of individual students in order to acquire information about their troubles and mistakes and, on the other hand, about their success and good ideas. Notice that this organizational model is quite time consuming for the teacher.

- *Optional course of mathematics.* This course is designed for students of the last classes who want to prepare for school-leaving examination and university entrance examination in mathematics. There are typically eighteen students in this course. In the computer laboratory some students work in couples on one computer. Three two-hour lessons are dedicated to teaching probability theory together with introduction to programming in MuPAD. Thanks to the fact that more talented and more interested in mathematics students participate in this course, more emphasis can be given to the optional topics. Students go through the parts comprising conditional probability (without the Bayes' theorem) and random variables (without the olympiad-level problems) but skip topics on random walks.
- *Mathematical seminar.* This seminar is designed for participants in mathematical contests and students who are seriously interested in mathematics. In contrast to the optional course of mathematics students attended this seminar in their free time. Content of this seminar is aimed especially as preparation for mathematical competitions. At the time of the experiment this group consisted of five talented students. (Two of them participated in the international physical olympiad and most of them usually participated in national mathematical olympiad.) Concerning the theory, conditional probability, random variable and random walks were revisited. More advanced topics such as the Bayes' theorem and the inclusion-exclusion principle were discussed and illustrated with olympiad-level problems. A very small number of students in this group allows them to discuss all their ideas with the teacher and classmates and to compare their solutions. Because of students' interest in programming, usually about a half of the lesson was dedicated to creating algorithms in MuPAD. It was very stimulating for the teacher that based on this experience one student also participated at programming category of the mathematical olympiad.
- *Optional course of informatics.* This course is intended for students of the last classes who wanted to study computer science or who were simply interested in computers and programming. There are usually fifteen students in this course. Taking into account that students are typically beginners in the computer programming, the teacher starts with the preliminary course in basics of algorithmization. It consisted of five lessons of programming in the *Karel language*, which is in authors' opinion optimal for the beginners. Then eight lessons of MuPAD programming followed. Within these lessons only small amount of necessary probability theory was explained, because main emphasis was devoted to the algorithmization. For this reason most of the probability problems were solved only by the computer simulation. During the course students became familiar with basic programming structures (`for, while` and `repeat` loop, if and `case` branching statements, procedures and functions) and basic data structures (numerical variables, arrays and strings). After MuPAD language course, Pascal language course follows and concentrates on additional programmers' acquirements such as manipulating files, controlling input and output, computer graphics etc.

During these "educational experiments" increase in students' activity and interest was observed. Students mastered MuPAD environment very quickly especially thanks to its simplicity. They successfully solved most of the exercises and homeworks. On the other hand, it should be emphasized that teaching by those methods have much higher demands on the teacher's preliminary preparation than usual.

6 Conditional probability – an example

In Subsection 4.6 we have discussed the most important concepts of our approach concerning the conditional probability and independence. To get a flavour of our examples and solutions, we include an example about the eccentric warden together with the solution using both the simulations and probability considerations.

Prisoner's problem[1]**.** In a prison there is a prisoner sentenced to death. Fortunately, the eccentric warden offers the prisoner a chance to live. The warden gives the prisoner 12 black balls and 12 white balls, two urns and tells him that tomorrow the executioner will randomly choose an urn and will draw one ball at random from it. If a white ball is drawn, the prisoner will be set free, if a black ball is drawn, the sentence will be carried out. How should the prisoner arrange the balls in the urns to maximize his chance for freedom?

Computer simulation: Following algorithm simulates a random choice of a urn and a random draw of a ball from the chosen one:

```
>> k:=7:                             // number of balls in the first urn
   i := 4:                           // number of white balls in the first urn
   n := 100:                         // number of experiments
   p := 0:                           // frequency of lucky experiments
   for j from 1 to n do              // perform n experiments
     urn := random(1..2)();          // randomly choose an urn
     if urn = 1 then                 // first urn
       ball := random(1..k)();       // draw a ball at random
       if ball <= i then             // if the drawn ball is white then
         p := p+1:                   // increase number of lucky experiments
       end_if:
       else                          // second urn
         ball := random(1..24-k)(); // draw a ball at random
         if ball <= 12-i then        // if the drawn ball is white then
         p := p+1:                   // increase number of lucky experiments
       end_if:
     end_if:
   end_for:
   print(float(p/n), "rel. freq. of experiments within the prisoner is set free");

   0.56, "rel. freq. of experiments within the prisoner is freed"
```

Students have many possibilities to experiment with this algorithm. Thay can modify values of variables k and i and try to form the hypothesis for which values of these variables the prisoner has the greatest chance to be set free.

Theoretical solution: Assume that no urn is empty. If the prisoner puts for example k balls into the first urn from which i are white, then the second urn has to contain $24-k$ balls from which $12-i$ are white. We denote by A an event "the drawn ball is white", by U_1 an event "the executioner chooses the first urn" and by U_2 an event "the executioner chooses the second urn". Since U_2 is a "complement" of U_1, we can apply the theorem of total probability to get

$$
\begin{aligned}
P(A) &= P(A|U_1)P(U_1) + P(A|U_2)P(U_2) \\
&= \frac{i}{k}P(U_1) + \frac{12-i}{24-k}P(U_2) = \frac{1}{2}\Big(\frac{i}{k} + \frac{12-i}{24-k}\Big) = \frac{12i - ki + 6k}{k(24-k)}.
\end{aligned}
$$

[1]This problem is taken from the book Anděl (2001) which can be recommended to students of secondary schools for additional study.

Let us denote $P(k,i) = P(A)$ to emphasize that $P(A)$ is a function of two variables, i.e. $k = 1, \dots, 12$ and $i = 0, 1, \dots, 12$. Now the teacher anounces the competition for students aimed at first finding the values k and i which maximize the function $P(k,i)$ and proving his/her result. Except the formal prove students can also use other arguments. They can, for example, count all possible values of $P(k,i)$ using MuPAD or spreadsheet, draw a 3D graph of this function in MuPAD etc. It is appropriate if all these possibilities are discussed and compared in the classroom together with formal proof that maximum of this function is $P(1,1) = 0.739$. Following algorithm counts and prints all $2 + \ldots + 13 = 90$ values of $P(k,i)$ that are of interest and find the greatest value between them and its "place":

```
>> DIGITS := 3:                     // No. of sig. digits in float. point numbers
   pmax := 0:                       // maximum value of probability P(A)
   for k from 1 to 12 do            // number of balls in the first urn
     for i from 0 to k do           // number of white balls in the first urn
       p := (12*i - k*i + 6*k)/(k*(24 - k)): // count probability P(A)
       print(NoNL, float(p), " "):  // print its value
       if p > pmax then             // look for the maximum value of P(A)
         pmax := p:                 // save this maximum value
         km := k:                   // save respective value of variable k
         im := i:                   // save respective value of variable i
       end_if:
     end_for:
     print():                       // print new line
   end_for:
   print(float(pmax), "maximum value of P(A)"):
   print(km, "value of variable k", im, "value of variable i"):
```

Despite the fact that above code looks very simple for experienced programmer, for most secondary school students it is an interesting exercise to write it down themselves.

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