

UNDERSTANDING OF ARITHMETIC MEAN

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The article deals with the experiences which learners have with the concept of average. This term is very common in their daily and school lives. The instruction of mathematics includes the arithmetic, geometric and harmonic mean as well as median and modus after data handling was introduced into the syllabus. Based on the sample of two different age groups of learners we were trying to find out what the understanding of the concept of arithmetic mean is, if students are more able to understand arithmetic mean as an algorithm or they can use it to solve an open ended task, and if there is a perceivable qualitative difference in the comprehension related to the learners' age. In accordance with the results we propose some guidelines in order to better understanding the concept of arithmetic and other means.

INTRODUCTION

The average of a set of data is probably the most common statistical concept encountered in everyday life. We read about average income, average monthly temperatures, average speed etc., consequently, we make different conclusions and decisions. We also hear phrases like an average student, an average family, students learn in average 3 hours per day ... The word average is in everyday life and in mathematics instruction usually used and understood as a synonym for arithmetic mean. Many studies have shown the teaching-learning of this concept is apparently easy, but the understanding causes difficulties.

The results of studies dealing with this topic show limited conceptual knowledge of the concept of average or arithmetic mean (Watson, 2006). There are indications that understanding of arithmetic mean increases with students' age. Most students seem to know the computational algorithm and manage problems with arithmetic mean by applying algebraic skills. Students are more able to find results with 'adding all and dividing', but are less successful in the reverse operation when they have to find an unknown value in a set of data where the average is given. The common conclusion of studies points out that the concepts of means are taught as rules or computational algorithms, while their learning situations related to understanding the concepts of average in appropriate everyday context are less frequent. The studies (Cruz, 2006; Mousoulides, 2006) emphasize the importance of solving open problems and suggest that some type of open questions can be used to examine students' ideas about the concept of arithmetic mean. Arithmetic mean and the concept of average can be better understood when contrasted with other means, median and modus.

We wanted to examine and compare the above findings with those in a sample of Slovenian students. We focused on knowledge of the concept and computational rule for arithmetic mean, conceptual understanding of arithmetic mean, distinguishing among means, and on how the concept of average is understood in the context of a given problem.

PROCEDURE, RESULTS AND DISCUSSION

In our case study 27 eighteen year old students in the final year of secondary school completed 10-item test and 20 thirteen year old students in elementary school completed 7-item test. The 7-item test is a reduced version of 10-item test. Both groups were taught data handling from primary level continuously as the curriculum prescribes at least 78 hours of instruction in elementary school and additional 10 hours in secondary school.

The first two exercises assess availability of concepts and procedures for average, arithmetic mean, median and modus. In the second exercise the concepts of frequency distribution and weighted arithmetic mean are included.

1. *The following data are given: 20, 20, 21, 23, 25, 30, and 31. Calculate arithmetic mean, median and modus.*

2. Marko threw the dice 40 times. The frequency distribution of number of dots is in the table below. What is the average number of dots in 40 throws?

Number of dots	1	2	3	4	5	6
Number of throws	7	6	8	5	8	6

The question in the first exercise was answered correctly by 80 % of the younger students and 76% of the older students. The students seemed to know the algorithm for arithmetic mean and basic rules for median and modus. The wrong answers were due to computational mistakes, some students confused modus (the most common value) and median (the middle value), which are in the Slovenian language also very similar words. The failure to distinguish between median and modus was more frequent in the older group, because the younger group had just learned about the concept of means. Some students replaced arithmetic mean with the nearest given data – this was probably due to their experience in assessing the final grade by frequently using arithmetic mean, which is replaced by the nearest integer estimation.

In the second exercise the students of both groups understood the word average as arithmetic mean. It was correctly solved by 35 % of the younger group students and 48 % of the older group students. The problem was frequency distribution where two variables, number of dots and number of throws were given – the students had difficulties in understanding weighted arithmetic mean. The results were worse despite the doubly underlined number of all throws. The variable number of dots caused an incorrect arithmetic mean of dots (1), while the variable number of throws caused an incorrect arithmetic mean of throws in this context (2).

$$\frac{1+2+3+4+5+6}{6} = 3,5 \quad (1)$$

$$\frac{7+6+8+5+8+6}{6} \quad (2)$$

In the wrong answers were also errors like (3), where number 6 is taken for the number of all throws instead of 40. Here are at least two possibilities for the students' decision, 6 possible dots or 6 regions in the given frequency distribution table.

$$\frac{1 \cdot 7 + 2 \cdot 6 + 3 \cdot 8 + 4 \cdot 5 + 5 \cdot 8 + 6 \cdot 6}{6} \quad (3)$$

Both groups of students had had experiences with calculating the average class grade for math tests.

The following task was given only to the older group.

3. Arithmetic mean of the numbers 77, 77, 42, 112, 140, 91 and x is equal 89. Calculate number x .

More than half of students (56 %) were able to find the unknown value x . Some students (15 %) made computational mistakes in solving the linear equation. The remaining students made different conceptual mistakes, like dividing by a wrong number (6 instead of 7), using the following reasoning (4):

$$(448 + x) : 2 \quad (4),$$

the sum of a given number is one number and the unknown x is another number. Then the solutions like $x = 1.16$; $x = -64.3$; $x = 270 \dots$ showed the students' inability to foresee the result.

The next two examples assess the ability to relate the numbers and corresponding result dynamically.

4. How does arithmetic mean change if all numbers are increased by 1?
 5. How does arithmetic mean change if one number is increased by 10?

The problems can be solved by the means of algebra mechanically, by reasoning on the basis of understanding the concept, or using the method of an example (one or more and make inductive reasoning).

The first problem was presented only to the older group. The agreed answer was that the change of numbers caused the change of result. There were 15 % incomplete answers in which explanations used words 'will change' and more precise 'will increase'. One answer was 'it increases for $\frac{1}{2}$ '. 63 % of the students answered correctly that the result is also increased by 1.

Interesting was the reasoning behind the result $\frac{1}{2}$, which seemed to be similar to the reasoning (4). These answers were not produced by the same student.

The task 5 is from mathematical point of view the same as the task 4, yet the students' performance showed the opposite – there were no correct exact answers in any of groups. The students felt the increase (60 % in the younger group and 59 % in the older group). Two students from the younger group thought that the result decreased. Then there were increases for 5 (9 answers in both groups, $4 + 5$). Number 5 was the result of dividing 10 by 2, which showed that they in everyday life probably operate with two numbers only.

Distinguishing between means (modus, median and arithmetic mean) and understanding means was evaluated by the following example.

6. *Luka collected the data about eye colour in his class. He found out that grey eyes are the most common in his class. What does this conclusion mean in the language of data handling?*

The task was presented to the older group only. 41 % of the students made the correct conclusion that Luka's finding corresponds to modus, 45 % of the students thought that the number of grey-eyed students is arithmetic mean, 11 % of the students mixed up modus and median, 13 % of the students did not respond.

The wrong answer 'arithmetic mean' also revealed unfamiliarity with the concept of variable (nominal and ordinal).

The following commonplace situation was investigated with an open question in both groups. The students faced the word average in a real-life context.

7. *The highest test score was 18 points. The classmates Tine and Luka reached an average of 10 points. What does the result tell you?*

We expected some detailed answers including a set of all possibilities, but there were only 4 correct answers in the older group, semantically and logically excellent descriptions in natural language, and none in the younger group of students. The correct answers were: at least one got a positive mark (3 answers), one got a negative mark, or both got positive mark (1 answer).

Among incomplete answers (41 %) were: they got 20 points together (2 answers), the arithmetic mean is 10 points (8 answers). One student transformed 10 points to 56 %; he assumed that the test had in total 20 points. The wrong answers (22 %) displayed common similar opinions: both got 10 points, both got a positive mark, they got around 10 points. Five students did not answer the question. There were no incomplete efforts in younger group. Some additional wrong answers appeared: they got mark 3, it was hard, each of them got 10 points, and each of them got average 10 points.

The word average seemed to be understood as arithmetic mean. Reverse mental operations caused difficulties to the students.

The following question concerned the concept of middle value in school context.

8. *Here are the test points for eleven students: 3, 20, 9, 1, 8, 12, 16, 14, 5, 5, and 10. Luka said: »I was somewhere in the middle. « What does this tell you? How many points did Luka get?*

To be 'somewhere in the middle' could be explained in three different ways: looking for a value in the middle of the ordered set, or assuming that 20 is the maximum number of points and the middle value is half of all, while the third possibility is explaining according to the frequency of points (which would be the grade 3).

As correct were considered the answers 9 and 9 or 10 (only 1 student), in the younger group 35 % of answers were correct, in the older 59 %. There were similar wrong answers in both groups: arithmetic mean 9.36, median 12 in the unordered set of data, 14 points (in our schools this used to mean grade 3 if 20 was total number of points).

For the last two tasks we took at first sight two similar problems from physics.

9. *The car is driving for one hour at the speed of 100 km/h and one hour at the speed of 120 km/h. What is its average speed?*

10. *The car is driving from the place A to the place B at the speed of 100 km/h and back from B to A at the speed of 120 km/h. What is its average speed?*

The correct answer to the task 9 was produced by 65 % of the younger students and by 100 % of the older students. Task 10 was not solved correctly in any of the groups. Only one student in the older group hesitated and did not answer the question. The right answer is related to

the concept of average speed in physics. In the task 9 the concept of average speed coincides with the concept of arithmetic mean (equal times), in the task 10 it is the harmonic mean of velocities

$$\bar{v} = \frac{s_1+s_2}{t_1+t_2} = \frac{2s_1}{\frac{s_1}{v_1} + \frac{s_1}{v_2}} = \frac{2v_1v_2}{v_1+v_2} \text{ (equal distances).}$$

From these examples we concluded that the students were not instructed in connecting, comparing and contrasting problems and contents from different subject areas.

The average velocity (scalar speed) \bar{v} of an object moving through a displacement Δs during a time interval (Δt) is described by the formula: $\bar{v} = \frac{\Delta s}{\Delta t}$. For equal times we get the arithmetic mean of speeds.

$$\bar{v} = \frac{s_1+s_2+\dots+s_n}{t_1+t_1+\dots+t_n} = \frac{vt_1+vt_1+\dots+vt_1}{nt_1} = \frac{v_1+v_2+\dots+v_n}{n}$$

CONCLUSION

In the older group of students the study assessed basic facts about arithmetic mean in contrast to modus and median in mathematical contexts (tasks 1 to 5), in everyday life (6 to 8), and in connection to physics (9 to 10). The low results in the tasks 5 and 7 were very surprising.

In the younger group the study also assessed knowledge in mathematical context (tasks 1, 2, 5), in everyday life (7, 8), and in connection with physics (9 to 10). When we compared the achievements with the older group, there was a surprisingly small difference in view of their different experiences.

The analysis presents the students' experience with the concept of average, which is very commonly used in their everyday life. Although data handling has become a part of the curriculum, there are deficiencies in understanding basic concepts, like median, modus, and arithmetic mean. The sample of two groups of different ages gave us quantitative and qualitative differences in understanding, but not to the expected degree. The students master computational algorithms for statistical means, yet they do not apply them to explain everyday situations.

The school practice should pay more attention to reverse processes, to understanding what the average tells us about the observed sample, to interpretations and explanations. Students are in the process of teaching and learning rarely faced with the following situations:

- Open-ended tasks in mathematical and everyday context, like: What happens with means (arithmetic mean, modus and median) if one number is changed or added?
- Complex statistical investigations, like: Use the reaction times (data table provided) and rank the four drugs by their effectiveness. Write a letter, explaining and documenting your results to the chairman of the Drug Industries Association (Mousoulides, 2006, p. 4-204).
- Reverse questions in mathematical and everyday context: The arithmetic mean of two numbers below 15 is 10. What are these two numbers?
- Questions fostering reasoning and decision making, including explanation: The average temperature at the place B is 20°C. Is place B good choice for summer holidays if you want to swim in a warm lake every day?

We can improve students' performance with the development of metacognitive strategies by asking questions to clarify exactly what students are trying to do or say, and we can make students more aware of the mental processes they use.

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