CONFIDENCE INTERVALS USING INTERVAL ARITHMETIC

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This paper deals with the surprising and interesting question posed by some students about constructing confidence intervals for mean differences in independent samples. The question was: is it possible to construct a confidence interval for the difference between two means by subtracting the intervals (an arithmetic interval operation) obtained for each parameter? Several activities created by the teacher in order to obtain an answer to this question are presented. Some other confidence intervals obtained via arithmetic interval are studied.

INTRODUCTION

Most teachers encourage their students to ask questions in the classroom. There are even some teachers, such as the present authors, who "reward" those students who do ask questions and actively participate in class. Therefore, some students occasionally formulate very interesting, significative questions, as is the case when a scientific genius enters the scene every now and them. Fortunately, sometimes these questions compel teachers to think, thus generating really interesting, worthwhile discussions among students. Such a case occurred to one of the authors of this paper, while explaining confidence intervals for the difference of means with independent samples to students in a first semester statistics course of Civil Engineering. A student asked the following question: Is it possible to obtain a 0.95 confidence interval for $\mu_1 - \mu_2$ by subtracting a 0.95 confidence interval for μ_2 from a 0.95 confidence interval for μ_1? Since this question was not immediately answerable, the teacher decided to take advantage of this unexpected topic by distributing *students* into *small groups* in order to discuss the question through activity-based statistics. This didactic method has a recognized pedagogical value in the teaching of Statistics (Sheaffer et al., 1997).

The use of small groups allowed the study of a large amount of confidence intervals for different parameters. They learned how to find the coverage probabilities and expected length of the proposed intervals, using simulation and some theorems learned in previous lessons. Students were able to obtain an increased understanding of the concept of confidence intervals and an improved knowledge of the learning objectives as described in chapter 10 of (Montgomery et al., 2003). The next sections of this paper will describe those activities carried out in classroom in order to answer the posed question. From a practical standpoint, the answer to these activities is related to the question if the confidence intervals do overlap, does that mean there is a no statistically significant difference between the groups? (Julious, 2004). The principal learning objective of these activities is encourages students to stretch their ways of thinking and not just learn methods handed down by authorities. Thus, the authors think to repeat these activities in future courses.

INTERVAL ARITHMETIC AND CONFIDENCE INTERVALS FOR MEANS DIFFERENCES

In order to address the original problem in the setting of interval arithmetic it was first necessary to familiarize students with this issue, and especially with the subtraction between two intervals. According to arithmetic interval, the subtraction between two intervals is defined by

$$[x_1, x_2] - [y_1, y_2] = [x_1 - y_2, x_2 - y_1]. \qquad (1)$$

Throughout this paper, the interval obtained from subtracting two intervals will be termed *difference interval*.

The confidence interval for the difference of means, both with known and unknown population variances were studied in this order. Thus, problems and activities were in increasing order of difficulty presented and performed. In both cases, for the sake of simplicity, normality of populations, homogeneity of variances and equality of sample sizes were assumed.

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Case 1: Known population variance

When dealing with the difference of means of independent samples of size *n* with known common variance σ^2, *the difference interval* obtained from *subtracting two confidence intervals* (according to Equation (1)) for μ_1 and μ_2, both with the same confidence level 1- *γ*, is given by

$$(\bar{x}_1 - \bar{x}_2) \pm 2 \cdot z_{\gamma/2} \cdot \frac{\sigma}{\sqrt{n}} \qquad (2)$$

where $z_{\gamma/2}$ is the value *z* such that $P(Z > z_{\gamma/2}) = \gamma/2$.

Question1: For any fixed *1-γ*, which is the coverage probability 1-*α* of the interval (2)? Once 1-*α* is given, is there a *1-γ* value such that the coverage probability of the interval (2) is 1-*α*?

If we recall that $\bar{X}_1 - \bar{X}_2$ has a normal distribution with mean $\mu_1 - \mu_2$ and standard deviation $\sqrt{\frac{2}{n}}\sigma$, then we can confirm that the confidence level 1-α of the interval (2) for the parameter $\mu_1 - \mu_2$ is $1-\alpha = 1 - 2P\left(Z > \sqrt{2} \cdot z_{\gamma/2}\right)$. Thus, if the intervals to be subtracted have the same confidence level 0.95, i.e., 1-*γ* = 0.95, then the *difference interval* (2) has a confidence level 1-*α* = 0.9944. Conversely, if we want the *difference interval* (2) to have a confidence level of exactly 0.95, then each one of intervals to be subtracted must have a confidence level $1-\gamma = 0.8342237$. It should be noted that this confidence interval agrees exactly with the confidence interval $(\bar{x}_1 - \bar{x}_2) \pm \sqrt{2} \cdot z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$, at the 0.95 confidence level, which is the exact interval for this case. In table 1 we show the correspondence between 1-*α* and 1-*γ* for various values.

Table 1. Correspondence between 1-α and 1-γ

1-γ	1-α
0.95	0.9944
0.90	0.9799
0.8342	0.95
0.7552	0.90

Case 2: Unknown Population variance.

In this case, the *difference interval* resulting from *subtracting two confidence intervals* for μ_1 and μ_2, both with the same confidence level $1-\gamma$, is $(\bar{x}_1 - \bar{x}_2) \pm \frac{t_{n-1,\gamma/2}}{\sqrt{n}}\left(s_1 + s_2\right)$. After computing several *difference intervals* for different values $1-\gamma$ and comparing them with the exact interval $(\bar{x}_1 - \bar{x}_2) \pm \frac{t_{2(n-1),\gamma/2}}{\sqrt{n}} \cdot \sqrt{s_1^2 + s_2^2}$ with the same confidence level $1-\gamma$, we could confirm, at least empirically, that the *difference interval* contains the exact interval. And this result led to the second question.

Question 2: Let a fixed value $1-\gamma$ be given. Is the exact interval contained in the *difference interval*?

This question was easy to answer since, from $t_{n-1,\gamma/2} \geq t_{2(n-1),\gamma/2}$ and $s_1 + s_2 \geq \sqrt{s_1^2 + s_2^2}$ it follows that $t_{2(n-1),\gamma/2}\sqrt{s_1^2 + s_2^2} \leq t_{n-1,\gamma/2}\left(s_1 + s_2\right)$. The answer to question two also allowed us to conclude that, if the intervals for μ_1 and μ_2 are constructed with the same confidence level $1-\gamma$ then the coverage probability $1-\alpha$ of the *difference interval* is greater than or equal to $1-\gamma$. This opens the way for the next question.

Question 3: Is there some value $1-\gamma$ such that the *difference interval* has a pre-established confidence level $1-\alpha$? For instance: which would be the $1-\gamma$ confidence level of the intervals involved in subtraction in order that the *difference interval* has a 0.95 confidence level?

In order to answer this question, students needed first to understand the difference between concepts such as "coverage probability" and "confidence level or confidence coefficient". They were told that, in general, probability coverage is a function of unknown parameters (in this case, μ_1 and μ_2) and that the (coefficient or confidence level) is the infimum over all coverage probabilities. Afterwards, the students were asked to run several simulations for various parameter values but keeping the seed of the random number generator fixed. In all cases, the coverage probabilities were always constant so that it made sense to speak of confidence level for the *difference interval*. Since question 3 was really difficult to answer, the teacher decided to deal with an easier version.

Question 4: Is there some value $1-\gamma$ such that the expected length of the *difference interval* exactly equals to the expected length of $1-\alpha$ of the exact confidence interval?

A brief introduction to the concept of expected length of a confidence interval was given, and the students concluded that the variables $t_{n-1,\gamma/2}\left(s_1+s_2\right)$ and $t_{2(n-1),\alpha/2}\sqrt{s_1^2+s_2^2}$ must have the same average. Given that, $t_{n-1,\gamma/2}$ and $t_{2(n-1),\alpha/2}$ are constants, a question remained without answer: What is the expected value of the random variables $\left(s_1+s_2\right)$ and $\sqrt{s_1^2+s_2^2}$? Students were taught the theoretical background showing that $\frac{s^2(n-1)}{\sigma^2}$ follows a chi-squared distribution with *n*-1 degrees of freedom, but, which is the distribution of the square root of a chi-squared variable? Or, which is its expected value? Students had access to a computer and internet in the classroom, and a search was performed where, by chance, (http://en.wikipedia.org/wiki/Chi_distribution) material on the square root of a chi-squared distribution was available. A happy coincidence! With this information, and given that s_1 is independent of s_2 and both have the same "chi distribution" with *n*-1 degrees of freedom, we can compute $E\left[s_1+s_2\right]$ without difficulty. On the other hand, since the variable $\frac{n-1}{\sigma^2}\left(s_1^2+s_2^2\right)$ has a chi-squared distribution with 2(*n*-1) degrees of freedom, then its square root has a chi distribution with 2(*n*-1) degrees of freedom, so we can also compute $E\left[\sqrt{s_1^2+s_2^2}\right]$. Finally,

$$t_{n-1,\gamma/2} = t_{2(n-1),\alpha/2}\frac{E\left[\sqrt{s_1^2+s_2^2}\right]}{E\left[s_1+s_2\right]} = t_{2(n-1),\alpha/2}\frac{\Gamma\left(\frac{n-1}{2}\right)\cdot\Gamma\left(n-1/2\right)}{2\cdot\Gamma\left(n-1\right)\cdot\Gamma\left(\frac{n}{2}\right)}.$$

In table 2, and for *n* = 5, we see that if the confidence level of the difference interval is equal to 1-*γ* = 0.83, then its expected length is equal to the expected length of the exact interval with confidence level 1-*α* = 0.95.

Given that an answer to question 3 was not immediately available; the teacher asked the students whether the answer to question 4 could also provide a solution to question 3. Therefore we asked the students if they could verify it based on a simulation study. Then they performed 50 estimates of the coverage probability for 10000 *difference interval* with 1-*γ* = 0.8320 and *n* = 5. The 0.95 confidence interval for these 50 values was [0.9484, 0.9495]. *This result was considered conclusive evidence that the confidence level of the difference interval was not* 0.95. One of the students observed that these values were practically the same, which led to some interesting comments on statistical significance and practical significance. The same chain of events occurred for the cases covered in table 2. The conclusion was obvious: if expected lengths are equal, the *difference interval* has a confidence level slightly smaller than the exact one. On the other hand, if we could manage to observe equal confidence levels, then the expected length of the *difference interval* would be slightly greater than the exact one.

Table 2. Values 1-α and 1-γ with n=5, 10, 20 y 50 such that the "difference interval" and the exact interval has the identical expected length

n = 5		n = 10		n = 20		n = 50	
1-γ	1-α	1-γ	1-α	1-γ	1-α	1-γ	1-α
0.95	0.9948	0.95	0.9945	0.95	0.9944	0.95	0.9944
0.90	0.9808	0.90	0.9802	0.90	0.9800	0.90	0.9800
0.8320	0.95	0.8337	0.95	0.8341	0.95	0.8342	0.95
0.7534	0.90	0.7548	0.90	0.7551	0.90	0.7552	0.90

INTERVAL ARITHMETIC AND CONFIDENCE INTERVALS

After these earlier cases were studied, we moved on to unknown and more imaginative scenarios. The teacher asked the students whether arithmetic interval could be used to compute confidence intervals for other parameters, such as the ratio of the two means, the percentile of a normal distribution or the variation coefficient. Within this group, we chose to focus on percentiles, specifically the 95th percentile of a normal population with mean μ and standard deviation σ. If μ and σ are know, the 95th percentile is μ + 1.644854 σ. For this parameter and the remaining percentiles from a normal population the literature offers different confidence intervals (Chakkaborti & Li, 2007). The resulting confidence interval obtained by applying arithmetic interval is obtained by computing two confidence intervals [a, b] for μ, [c, d] for σ, and then performing the operation [a + 1.644854c, b + 1.644854d]. In order to study the properties of the coverage of the interval we carried out an additional simulation study for different sample sizes. Our conclusions were: for n = 5, if each one of the intervals [a, b] and [c, d] is generated with a confidence level 1-γ = 0.8485, then the combined confidence level of the interval is approximately 0.95. Also, the confidence level was practically unnoticeable for a large sample sizes.

Although it may be bold to say so, this unorthodox methodology can serve as a technique to compute confidence intervals where a proposed agreement under rigorous statistical methods is not available.

CONCLUSIONS

In this paper, we present a series of statistical activities designed to answer an intelligent and rather unusual question from a student during a lecture. The designed activities helped students extend their understanding of what a confidence interval is, and how to compare its two main properties: confidence level and expected length. In addition to enhancing their understanding of these concepts, the students learned as well how to perform a simulation study. Teachers do need to pay particular attention to questions in the classroom, and not to discard them, because they can provide excellent opportunities to motivate students and to increase their understanding of the subject at hand. Sometimes, as is the case in this communication, a new methodology may result, offering new solutions to old problems. Class sections following these activities were worthwhile and stimulating for both teacher and students: a noticeable increase in student participation was observed in their answers to the teacher questions, or in their own questions for the teacher.

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