THE TOTAL PROBABILITY THEOREM, THE "DEPENDS" ARGUMENT AND SIMULATION

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As an alternative to the Total Probability Theorem, the "depends" argument that students use to calculate marginal probabilities is studied. We discuss an experience with undergraduate engineering students who took a computed aided basic probability course based on the frequency approach. The result of this experience shows that an adequate interpretation of the outcomes of simulated random experiments allows conjecturing and arguing algebraic results of the theory of probability.

INTRODUCTION

The conceptions and the difficulties that students have with conditional probability have been widely studied for a long time (Kahneman *et al*., 1982; Falk, 1979, 1986; Pollatsek *et al*., 1987; Gras and Totohasina, 1995).

A very common notion among students is that a conditional probability is a measurement of the causal relation that exists between two events, in such a way that the values that people assign to the conditional probabilities depend on the causal relation of the conditioned event in relation to the conditioning event (Kahneman *et al*., 1982).

Another deeply-rooted conception they have is the chronological conception by which it is thought that conditioning events are always thought to take place *before* conditioned ones, thus the latter are regarded as a sort of result of the former (Falk, 1979, 1986; Gras and Totohasina, 1995).

Based on this reasoning a research was carried out whose results are shown in this paper, examining how students deal with marginal probabilities, a subject that has not been studied enough from the point of view of the teaching of probability.

Following Falk (1986), who suggested performing an experiment to convince students of the errors in their misconceptions, a course on basic probability was given based on the frequency approach to probability, purposing to observe the modification these misconceptions would undergo.

THE STUDENTS AND THE METHODOLOGY

Six undergraduate engineering students took part in the research. The program was carried out in twelve two-hour sessions. The main topics in elementary probability theory were reviewed: the classical approach and the frequency approach to probability, the addition rule, independence of events, the law of large numbers, the product rule and the theorem of total probability. The software used was the *Fathom* package (Finzer *et al*., 2000). The methodology used was problem solving, in such a way that the search for solutions would allow them to conjecture the theoretical results suggested by these solutions. The general results can be seen in Yáñez (2003).

RESULTS AND ANALYSIS

After some sessions devoted to the basic results of the theory (addition rule, the law of great numbers, independence and the product rule, conditional probability and the general product rule), the Theorem of Total Probability was undertaken in the twelfth session. The first problem asked of them was Falk's ball problem:

An urn contains two white and two black balls. We shake the urn thoroughly and blindly draw out two balls, one after the other, without replacement.

(i) What is the probability that the second ball will be black?

All students said it depended on the color of the first drawn ball; as Laura said: *"It depends on the outcome of the first draw, because it's more likely that the second will be black if the first one was white, and vice versa."*

The use of the term "depends" reflects the *outcome approach* (Konold 1989), which, in this case, consists of considering only one trial of the random experiment and the probability value associated with this trial.

In order to confront this 'depends' strategy used by all students, they were invited to compare their answers with the results obtained from experiment and simulation, so as to find an algebraic expression that would account for these results. The activity was the following:

Form groups of two and do the experiment 100 times using the bags and balls given to you. Then collect the results from all pairs and answer the question again: What is the probability that the second ball drawn be black?

The experiment was done 300 times, in 149 of which the outcome of the second draw was black. If B_2 is the event of getting a black ball on the second draw, $\#B_2$ denotes the number of times this event is realised, and N is the overall number of trials, then:

$$P(B_2) \approx \frac{\# B_2}{N} = \frac{149}{300} \approx 0.496.$$

This prompted the following discussion:

Researcher: *Laura, what can we tell from the experimental evidence?*
Laura: *The probability is one half. In half of the cases the second ball will be black; in the other half it will be white.*
R: *Then, the probability "depends"?*
Laura: *No.*
R: *Looking backwards, why did you think that the probability "depends"?*
Daniel: *Because we do not focus on cases, let's say, in the long run, we only focus……there are four balls there and I fix one of them, so now there are more on the other side, then the question comes up whether it……indeed depends, because there are more on the other side.*
R: *What does it mean 'in the long run'?*
Daniel: *It's after many cases that the probability becomes established.*

Daniel's argument suggests that the attempt to adopt the classical approach to calculate precisely the probabilities at stake may induce the outcome approach, and the rejection of the 'long run': *"we focus… there are four balls there and I fix one of them, so there are more now on the other side"*. The difficulty lies in the fact that students do not realize they are dealing with a composite problem and that the sample space is not related to the number of balls in the urn, but corresponds to a set of pairs formed by considering the number of balls present in the first and second draws. In fact, the translation of the results of experiment and simulation in terms of counting of the possibilities is a didactic challenge worth developing.

Another possible explanation is the influence of the time axis. The first draw that happens *before* and the second draw that happens *after* could lure students into these analyses that many of them would deem "logical."

Groups of two were formed again to work on the final activity, which asked them to devise a frequency argument that would allow them to calculate the required probability.

We cite herein an argument from a pair of students who arrived at the frequency justification that leads to the exact calculation for the probability that the second drawn ball be black. They took as reference the *Fathom*'s procedure that simulated the experiment that is the computer version of the tree diagram:

First Draw: RandomPick ("B","N")

Second Draw: $if\,(\Pr imera = B)\begin{cases} RandomPick("B","N","N") \\ RandomPick("B","B","N") \end{cases}$

Alfredo and Daniel: *We know that if the first ball was white, then the probability of the second being black is 2/3.*

$$\frac{2}{3} = \frac{\# N_2}{\# B_1}, \quad \# B_1 = \frac{M}{2} \quad \Rightarrow \quad \# N_2 = \frac{M}{3}.$$

We've solved part of the problem. Now if we know that if the first ball was black, then the probability of the second being black is

$$\frac{1}{3}=\frac{\#N_2}{\#N_1}, \quad \#N_1=\frac{M}{2} \quad \Rightarrow \quad \#N_2=\frac{M}{6};$$

$$\#(N_2)=\frac{M}{3}+\frac{M}{6}=\frac{M}{2}$$

Having done this we can now write the second formula, which is

$$\frac{\#(N_2)}{M}\approx\frac{M/2}{M/1}\approx\frac{M}{2M}=\frac{1}{2}.$$

They calculate the probability by finding the number of trials in which the second draw is black. To do this, they find out how many trials give white on the first draw and black on the second, and how many give both black, and then they add these results. Dividing the sum by the total number of trials (*M*) gives the answer they were looking for. So they relate classical probabilities (implied by the constitution of the urn itself) with the frequency of these events, omitting the fact that they are just making approximations.

ANALYSIS OF RESULTS

The "depends" strategy that students used to calculate this marginal probability, after an intensive course focused on the frequency approach, shows the difficulty that arises when contrasting the logical arguments, inherited from the classical approach, with the logic of random phenomena founded on the stability of outcomes in a high number of trials. In other words, we are dealing with an understanding of the law of great numbers.

The way Alfredo and Daniel obtain the correct probability by counting the cases supports the idea of using the frequency approach as a reference to show the meaning and truth of the algebraic equalities. This strategy, which we could call the *restricted frequency approach* (any finite number of cases is enough) reminds us about the results of Gigerenzer and Hoffrage (1995), who recommend the use of frequential information as a mechanism to solve conditional probability problems more easily.

Though the students were not asked to do it, it is clear that if, instead of the specific probabilities implied by the urn's composition the theoretical values had been adopted, following Alfredo and Daniel's reasoning the Total Probability Theorem would have been obtained.

In any case, this procedure shows that the analysis of a particular example allows us to infer a general result that can be later justified by substituting the particular values of the problem for theoretical ones.

CONCLUSIONS

This experience demands our attention about various aspects of the concept of probability and its frequency interpretation: First, the 'depends' argument, used by students in place of the Total Probability theorem, at least when the factor of time is present in the formulation of the problem; second, the importance of experiment and simulation which allow getting results that could oppose the "logic" used by the students; finally, the possibility of turning to frequency reasoning, linked to the reality of results from experiment, that allows, through arithmetical considerations, conjecture algebraic formulas that accounts for these results.

We should add that it still needs to be found out whether or not the 'depends' argument is independent of the time axis. In the case of the Falk's ball problem, made up of two successive draws, this would amount to finding the students' answers if the draws are made simultaneously.

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