THE MEANING OF STATISTICS VARIATION IN UNIVERSITY BOOKS IN SPAIN

Antonio Estepa-Castro
University of Jaén, Spain
Juan Ortega-Moya
UNED, Spain
aestepa@ujaen.es

In this paper we fix the institutional reference meaning of variation and its measures in university books for the first university courses, using the six elements of meaning of the "ontologic-semiotic approach of mathematical cognition." The elements of meaning in books are identified. The deficiencies and possible difficulties that students can find, are considered. From the descriptive point of view, the complexity of topic variation and their measures is established. We conclude by pointing out the usefulness of the results.

INTRODUCTION

We are developing an educational research project on statistics variation and its measures in Secondary Education. The starting point is the study of the nature of variation, that is, the institutional reference meaning of variation and its measures in university books, which is a useful point of reference in order to characterize the institutional meaning intended in Secondary Education in Spain. In Mathematics Education, the study of the nature of Mathematics and the mathematical objects has been emphasized for a long time by such prestigious authors as Beagle: "One prerequisite to a study of the learning of mathematics is a clear understanding of the nature of the mathematics to be learning" (Beagle, 1979, p. 1).

THEORETICAL FRAMEWORK

Our theoretical framework is based upon "Theory of Semiotic Functions (TSF). An ontological-semiotic approach of mathematical cognition and instruction," developed by J. D. Godino *et al*. for more than ten years (Godino, 1996, 2002, 2003; Godino and Batanero, 1994, 1998, 1999; Godino and Recio, 1998). The TSF considers mathematics as a conceptual system, logically organized, whose parts are intimately interrelated. Mathematics is a human activity whose objective is to solve a series of problematic situations which are internal or external to mathematics itself. As a result of that activity mathematical objects emerge as entities that are used in the resolution of problematic situations. The mathematical problems and their solutions are shared by groups of people or institutions that become involved in their study.

This meaning of mathematical objects emerges from several practices (operative and discursive), accomplished when problems are solved. This meaning has a global character. Consequently, this meaning is divided into smaller units which are called meaning elements, namely, *situations* (more or less open problems, applications, activities that induce to mathematics activity); *actions* (operations, algorithms, procedures); *language* (words and representations as terms, formulations, equations, graphics, tables, ...); *concepts* (definitions and mathematical notions); *properties* (of the mathematical objects) and *arguments* (argumentation and reasoning that make the previous meaning elements correlate). Confrey's (1990) concept of conception is also used: "These categories of children's beliefs, theories, meanings, and explanations will form the basis of the use of the term *student conceptions*" (p. 4).

The main purpose is to study the meaning of statistics variation and its measures in university books. In the last few years papers have been written on statistical variation, from the cognitive point of view. Most of the papers presented in "The Third International Research Forum on Statistical Reasoning, Thinking and Literacy" (SRTL-3) were on statistical variation. Melitou (2002) offers an extensive bibliography on variation from the educational point of view. Reading and Shaughnessy (2004) make a summary of recent research into reasoning about variation.

SAMPLE

Fourteen books constitute the sample (Annexe I). Most of the books were chosen among the most widely used in the programs of several Statistics subjects in the first university courses

of 19 Spanish universities. Some were chosen because they were classic books or due to their authors' prestige. Most, although not all, are textbooks. The average number of pages devoted to the description of variability is 17.36 pages, with a standard deviation of 9.26 pages.

CONCEPT OF VARIATION

Variability has two meaning in the books studied: a) Variability is the variation/concentration of the data around an average (we found it in 13 of 14 books analysed). b) Variability is the variation/homogeneity of the data, from a datum to each other (we found it in 5 of 14 books analysed). We can distinguish two different conceptions, but when they are measured they are equivalent, since it is possible to demonstrate that the average of the square of all the differences between two data of a data set is equal to twice the variance of same data set. From a cognitive point of view, b) is more intuitive than a); consequently, the consideration of b) facilitated the beginnings of the conceptualization of variability.

In this study we considered the following descriptive measures of variability: range, interquartile range, average deviation, variance and variation coefficient. In the following section, we analyse the meaning elements of these measures.

MEANING ELEMENTS

Situations

S1. *Variation by ranges:* Finding the maximum difference in a set or subgroup of data.

S2. *Variation by deviations:* Measuring, in a suitable way, the variation of the data around an average.

S3. *Global comparisons.* Two kinds:

- S3a. *Comparing the variation of two or more distributions measured in the same magnitude.*
- S3b. *Comparing the variation of two or more distributions measured in a different magnitude.*

S4. *External local comparisons:* Comparing the relative position of data in different distributions.

S5. *Internal local comparisons:* Detecting and interpreting outliers.

S6. *Inverse problem to measure the dispersion:* Generating data with certain requirements on its variation.

S7. *Choice of the best measurement of variation:* Deciding the best measurement of dispersion in a data set.

Table 1: Frequency of the situations found and the books in which they appear

	SITUATIONS							
	S1	S2	S3a	S3b	S4	S5	S6	S7
Total number of situations	53	81	45	9	14	15	10	16
Total number of books	12	14	14	7	6	4	6	6

Observing Table 1, we can deduce, that, as for their use, three kinds of situations stand out: S1, S2 and S3a. These situations are used in all or almost all the books and the rest, except S5, approximately in half the books. The students who study in books where situations S3b, S4, S5, S6 and S7 are not used could acquire incomplete conceptions on variation. For example: the people who study in books where situation S3b (to compare distributions measured in different magnitude) or S7 (choice of the best measurement of dispersion) are not used, can acquire incorrect conceptions: distributions measured in the same magnitude can only be compared, in the first case; all measurement of dispersion is valid for all data set, in the second.

Actions

We found the following actions.

A1. *Calculating measures of spread of a series of data (without tabulation).*

A2. *Calculating measures of spread of tabulated data (not grouped).*

A3. *Calculating measures of spread of tabulated data, grouped in intervals.*
A4. *Calculating measures of spread with calculator and/or computer*.
A5. *Calculating measures of spread of transformed data (brief calculation).*
A6. *Calculating measures of spread of data displayed graphically*.
A7. *Weighed calculation of dispersion. Calculating measures of spread of separated data in subgroups*
A8. *Calculating measures of dispersion without the data (from information based on properties, definitions, formulae, etc.).*
A9. *Making inverse calculations with dispersion measures.*
A10. *Calculating and interpreting the interval* $(\bar{x} - k\sigma, \bar{x} + k\sigma)$ *and the percentage of data that it contains.*
A11. *Representing graphs that contain information on dispersion.*
A12. *Standardizing variables.*
A13. *Detecting dispersion by comparing or interpreting graphs.*

In Table 2 we can see the frequencies of actions and the number of books in which they appear. No book contains all the actions, which can make the learning obtained in these books on variation and its measures difficult.

Table 2: Frequency of actions found and the number of books in which they appear

	ACTIONS												
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13
R[1]	12/7	3/3	3/3	4/4	0/0	21/4	0/0	3/1	0/0	8/3	19/4	8/4	17/7
IR[2]	17/6	7/4	8/6	4/4	0/0		0/0	1/1	1/1				
AD[3]	1/1	5/4	4/4	2/1	0/0		0/0	0/0	0/0				
V[4]	25/8	11/5	12/10	11/6	2/2		1/1	3/2	5/2				
SD[5]	52/10	30/10	22/10	16/7	9/3		2/1	9/5	6/3				
VC[6]	8/4	15/6	6/6	4/1	0/0		0/0	6/2	0/0				
TO[7]	115	71	55	41	11	21	3	22	12	8	19	8	17

[1] Range ; [2] Interquartile range ; [3] average deviation ; [4] Variance ; [5] Standard deviation ;[6] variation coefficient ; [7] Total number of actions of that kind; x/y → x (actions of that kind)/ y (number of books in which they appear)

Language

- *Terms and expressions*: We found 43 terms and expressions used to refer to variation and its measures.
- *Symbols that represent the dispersion measures*: We found 6 symbols to represent each of the following measures: range, interquartile range and average deviation; 5 for variance and 4 for standard deviation and coefficient of variation.
- *Algebraic expressions (formulae)*: We found 7 different formulae for each of the following measures: range, interquartile range and average deviation; 19 for variance; 20 for standard deviation and 6 for coefficient of variation .
- *Graphs*: We found two kinds of graphs: a) statistical (bar graphs, histograms, box and whiskers... b) geometric, generally linear to explain the dispersion.
- *Tables*: We found two kinds of tables: a) Frequency tables, b) Tables to facilitate the calculations.

This diversity of elements of expression, representation and communication which conform the language makes the learning of the mathematical objects related to variation difficult.

Concepts. Definitions

Not all the definitions appear in all the books. We can see the frequency of the definitions found in Table 3.

Table 3: Frequency of the definitions found

CONCEPT	DEFINITION	F[1]
RANGE	D1.*The difference between maximum and minimum value.*	10
INTER QUARTILE RANGE	D2. *The difference between the third quartile and the first quartile.*	10
	D3. *The spread of the central half of the data.*	5
	D4. *The difference between 75th and 25th percentiles.*	3
AVERAGE DEVIATION	D5. *The average of the absolute deviations of the data with respect to the mean of that data set.*	6
VARIANCE	D6. *The average of the squared deviations of the observations from their mean.*	8
	D7. *The square of the standard deviation.*	8
	D8. $\sigma^2 = a_2 - a_1^2$, where a_n = nth raw moment (i.e., moment about zero)	6
STANDARD DEVIATION	D9. *The square root of the variance*	10
	D10. *Formula,* $\left(\left(\sum_i (x_i - \bar{x})^2 \cdot n_i\right)/N\right)^{1/2}$	9
	D11. *A certain kind of average.*	4
	D12. $\sigma = \sqrt{a_2 - a_1^2}$, where, a_n = nth raw moment (i.e., moment about zero)	2
COEFFICIENT OF VARIATION	D13. *The coefficient of variation of Pearson is the quotient between the standard deviation and the mean.*	6
	D14. $CV_{Pearson} = \sigma / \lvert\bar{x}\rvert$,	3
	D15. $CV = (\sigma / \bar{x}) \cdot 100$	2

[1] Frequency. In some cases the frequencies of a concept totalise more than 14 because there is more than one type of definition in a given book

Properties

Three kinds of properties are considered: numerical, algebraic and statistical. Not all the properties appear in all the books. In Table 4, we can see the frequencies of the properties found and the number of books in which they appear.

Arguments

The previous elements of meaning are linked and related by means of arguments and reasonings in the books analysed. The kinds of reasonings that we have found, can be classified in the following way: *a) Arguments based on properties of numbers and operations. (Facility and/or rapidity of calculation, division by zero...); b) Arguments through examples, counterexamples, verification of properties; c) Arguments supported by ostensive tools (graphics, tables, drawings...); d) Deductive verbal arguments, based on properties of dispersion measures; e) Deductive algebraic arguments, for example, demonstrations.*

DISCUSSION

Traditionally, the descriptive study of variation has been considered as a topic with few difficulties, both in its teaching and in its learning. Its enormous complexity has been stated through the presentation made from the data found in the books analysed on variation and its measures (due to length restrictions in this paper, a summary has just been presented; we will extend some of the sections in the presentation in ICOTS-7). This complexity can be deduced from the extensive and varied number of meaning elements found: different situations and actions; plenty of terms, symbols, formulae, tables and graphs; different definitions, many properties and diverse kinds of argumentations. Some elements of meaning do not appear in the books under study, which can make students acquire inadequate conceptions on variation and its measures.

Our results will be useful to fix the institutional reference meaning at university and for a forthcoming didactic study we are carrying out in Compulsory Secondary Education in Spain. It can also be useful to plan the teaching of this topic, since we have detected a considerable amount

of important elements of meaning, which may make their learning difficult. Finally, it can be useful in order to design and plan educational research into this topic.

Table 4: Frequency of properties and number of books in which they appear

PROPERTIES	F[1]	N[2]
PROPERTIES. Numerical properties (NP)		
NP1. *The ranges, the average deviation and the standard deviation are measured in the same units as the data. The variance is measured in square units of the data. The coefficient of variation is a-dimensional.*	41	12
NP2. *All measure of variation is essentially nonnegative.*	6	6
NP3. *If mean is zero the coefficient of variation does not exist.*	6	6
NP4. *The variance of n data (n>2) is bounded,* $\frac{d^2}{2n}+\frac{2}{n-2}a^2 \le s^2 \le \frac{d^2}{4}-a^2$, d *being the range and* a *the mean that would be obtained when changing the origin to the midpoint of the range*	2	2
NP5. *The standard deviation of n data is bounded both at the top and at the bottom*	2	2
PROPERTIES. Algebraic properties (AP)		
AP1. *The value of the dispersion measures depends on the form to group the data throughout the range.*	4	4
AP2. *If there is a linear transformation, the dispersion measures behave thus:* $R(ax_i + b) = \|a\|\cdot R(x_i)$; $IR(ax_i + b) = \|a\|\cdot IR(x_i)$; $AM(ax_i + b) = \|a\|\cdot AM(x_i)$; $V(ax_i + b) = a^2\cdot V(x_i)$; $S(ax_i + b) = \|a\|\cdot S(x_i)$	28	11
AP3. *The variance has more comfortable algebraic and arithmetical properties than the standard deviation and the average deviation.*	4	4
PROPERTIES. PE. Statistical properties (SP)		
SP1. *The sum of the deviations with respect to the mean is zero.*	10	10
SP2. *König' theorem (property of deviation):* $\sum(x_i-\bar{x})^2=\sum(x_i-a)^2-n(\bar{x}-a)^2, \forall a \in R$	3	3
SP3. *If all the observations are equal, the variation is null.*	2	2
SP4. *A greater value of the dispersion measures means greater dispersion.*	16	8
SP5. *All the values that the variable has take part in the calculation of the average deviation, variance, standard deviation and coefficient of variation take part. On the contrary, not all the data take part. in the calculation of the ranges.*	17	8
SP6. *The modification of some data and the addition or the elimination of some data produce changes in the average deviation, variance, standard deviation and coefficient of variation; nevertheless, this cannot influence on the ranges (Robustness).*	4	4
SP7. *Each dispersion measurement is the optimal one with respect to some criterion of kindness related to the search of the central value. Thus, the variance minimizes the quadratic average of the deviations with respect to the mean; the average deviation with respect to the median minimizes the average of the absolute deviations with respect to the median; the semi-ranges minimize the greater possible deviation with respect to the centre.*	9	6
SP8. *The ranges measure the dispersion without reference to averages. The standard deviation and the variance measure the dispersion of the observations around the mean.*	7	3
SP9. *The coefficient of variation is useful to compare the variation of variables measured in different units or that take measured values in different magnitudes*	4	4
SP10. *If the mean as measurement of central tendency is not used, the standard deviation and the variance as dispersion measures are not used either. This is especially certain in the skewed distributions, in which it is preferable to use the median and the interquartile range.*	3	3
SP11. *Twice the variance equals the average of the quadratic deviations of each element to all the others of the distribution, including itself.*	4	4
SP12. *The standard deviation of a standardized data set is one.*	4	4
SP13. *The inequality of Tchebychev (* $fr(\|x_i-\bar{x}\| \ge ks) \ge 1-(1/k^2)$ *) is true for any distribution.*	4	4
[1] Frequency. [2] Number of books where the property appears.		

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ANNEXE I. BOOKS ANALYSED

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