ANALYSING TEACHING AND LEARNING PROCESS FOR THE LAW OF LARGE NUMBERS: IMPLICATIONS OF USING SOFTWARE IN TEACHERS EDUCATION

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In this paper we analyse an intuitive approach to the study of the empirical law of large numbers by a pair of student teachers. The learning is based on the use of a random experiment simulation applet with feedback by a lecturer. The analysis is based on some theoretical tools taken from the onto-semiotic approach to mathematical cognition and instruction (Godino, 2002). In particular we assess the epistemic, cognitive and instructional suitability of the study process. We deduce some requirements of the simulation device characteristics and the lecturer's role to increase the suitability of the teaching and learning process.

CONTEXT AND RESEARCH PROBLEM

This research follows previous teaching experiences carried out in the context of primary school student teachers preparation and were based on computer resources ("Edumat-Maestros" Project, Godino *et al*. 2004). These experiences, all of them related to specific mathematical contents, were implemented with several groups of students in three complementary scenarios:

- Scenario 1: traditional classroom (blackboard, textbook and computer display).
- Scenario 2: computer lab; students working alone or in pairs with some applets, assisted by a lecturer.
- Scenario 3: clinical monitoring of a pair of students interacting with computer programs supported by a lecturer.

In this paper we will use the information collected in scenario 3, for the topic "empirical law of large numbers," using the NCTM "box model" simulation software, available at: http://illuminations.nctm.org/imath/6-8/BoxModel/index.html. The research questions were the followings: a) What did the students who followed this study process learn? b) What factors did condition their learning? and c) How could the suitability of the study process be improved?

Although this is a very specific experience, the facts observed and their interpretation, with the help of the onto-semiotic approach to mathematical knowledge (Godino, 2002; Godino, Batanero and Roa, 2005) will allow us to identify some general cognitive and instructional phenomena. This case study is also a methodological example of didactical analysis based on this theoretical framework.

Starting from the transcription of the activity of one pair of students interacting under guidance from a lecture we will describe the study process as a stochastic process with different trajectories (Godino, Contreras and Font, 2004): epistemic (implemented institutional knowledge), educational (lecturer's role and students' tasks), mediational (use of technological resources) and cognitive (students' knowledge). The analysis will be focused on identifying the competences achieved by students as well as their epistemic, cognitive and instructional conflicts. We will begin by making explicit the epistemic referential knowledge related to the law of large numbers and associated notions. As a result we will identify the crucial role of the lecturer in optimising the teaching and learning process and selecting appropriate technical resources.

THE STUDY PROCESS

Students were given a written "practice guide" where the applet and the activities proposed were described in order to help them carry out the practical activities in the computer lab. We used the "box model" random simulator, which is available from the NCTM web site mentioned above. The aims of the practical activities were as follows:

a) Simulating random situations in a computer setting.

b) Studying and analysing the simulations results.
c) Comparing initial prediction to empirical (simulated) results
d) Getting acquainted with probabilistic elementary concepts and language.

The applet is composed of two modules: One module serves to simulate a series of simple experiments, such as throwing coins or dice, represent the relative frequencies distributions and compare them to the probabilities of the events. The second module allows us to simulate series of compound experiments, such as throwing several "dice" and to study the distribution of the sum and mean of the results. We will only consider the first module in this paper, due to limitation of space. The applet used in the first module is described as follows:

Applet Description (Figure 1a):

- Data input: To set the urn composition, we select some numbered "balls" that are introduced in the urn, from a set of numbers and with the possibility of repeating a ball, to introduce events which are not equally likely.
- Simulation: By clicking the Start key, a ball is randomly drawn from the box (with replacement) and we can observe, in real time, the relative frequencies of observing a specific event as the number of draws increases. Since the device only allows sampling with replacement, we only can simulate independent random experiments.
- Stopping the simulation: By clicking the Pause key the process stops and by clicking on the bar chart the relative frequency for each event is shown. We can move a slide bar to observe the results obtained.

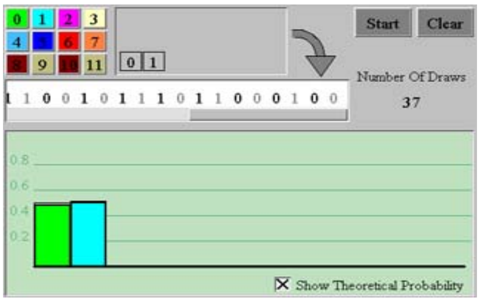

Figure 1a: Box model simulator

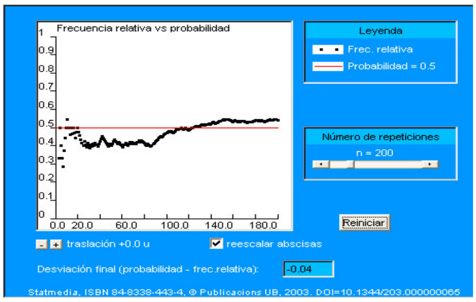

Figure 1b: Throwing a coin with Statmedia

Below we describe the questions posed in the "practice notebook" to students, about this module:

> *Produce 10 simulations of the tossing of a coin (introduce the numbers 0 (head) and 1 (tail) in the box) and click Pause after 10 tosses. Observe the behaviour of the relative frequency for each event and compare them with the theoretical probability.*
>
> *a) What is the relative frequency of heads after 10 draws?*
>
> *b) Why is it different from the probability? Continue the simulation until 50 draws.*
>
> *c) Are the differences between relative frequencies and probabilities of getting head and tail smaller now smaller? Explain why this happens.*

INTENDED STOCHASTIC KNOWLEDGE

The "box model" applet serves to introduce the study of the "empirical large number law" in an informal way, by producing a series of relative frequencies from increasing numbers of experiments. "*The progressive stabilitization of the relative frequency of a given outcome in a large number of trials, that has been observed for centuries and was translated by Bernoulli to a mathematical theorem, served as a justification for the frequentist definition of probability*".… "*This idea again is not free of difficulties, because the specific nature of random convergence is difficult to grasp and long runs, coincidences, or unexpected pattern are counterintuitive*" (Batanero *et al*., 2005, p. 30)

We can describe the problem-situation giving rise to this law in this way. How do relative frequencies of an event behave when the number of experiments increases indefinitely? When we throw a coin, for example, we cannot foresee if we will get head or tail; if we throw the coin 10 times, we expect a relative frequency around 0.5, but we do not know the exact value. However, it is mathematically established that the relative frequency of getting a head, after throwing a fair coin n times, f_n, is closer to ½ as n increases indefinitely. But the "convergence" of this sequence of values is not of an analytical convergence of a real number sequence; it is a "stochastic convergence," which means that the deviation between f_n and p, for a fixed n, can be greater than a fixed value. "The probability of getting a given deviation after n throws decrease as n increases, that is, $\Pr(f_n - p < \varepsilon)$ tends to 0 when n tends to infinity."

Moreover, this stochastic convergence can be slow, showing fluctuations and relatively large runs. These features can be explored using simulations of random phenomena, in order to build appropriated intuitions thereof.

THE TEACHING AND LEARNING PROCESS

In this section we describe the use of the simulation applet in the teaching and learning experience, the knowledge involved in the tasks carried out (sequence of implemented epistemic configurations) and some aspects of the instructional and cognitive trajectories.

Implemented Epistemic Trajectory

For 45 minutes, students interacted with the applet, used the two modules and simulated several experiments with coins and dice. We will mainly analyse the simulation of the tossing of a coin from the first module. In the "theoretical class" (teaching setting type 1) the lecturer had previously introduced the basic notions on random experiments (events, probability, Laplace's rule) and solved some elementary probabilistic problems.

The lecturer briefly explained the simulation applet and asked the students to reflect and express their ideas. Students started by simulating 50 draws of a coin; they expected to observe "the stability of the relative frequencies" around 0.5, the probability value for the events under the hypothesis that the simulation was unbiased. This hypothesis was not discussed. The simulator shows in the bar chart the value of "theoretical probability" for each event. Student S1 described this value as "*the central value that it should appear.*" Students stopped the experiment after 50 throws and observed that the relative frequencies were close to the probabilities (about 0.5).

After simulating the throwing of several dice (with 6, 12, 7 and 4 possible results, and always with small series of simulations) students repeated again the throwing of a coin 50 times. This time they did not obtain the expected results. After 50 draws the relative frequencies were 0.36 and 0.64, very different from the probabilities. Students did not try to increase the number of replications and they moved to explore the second applet module. They were very "convinced" that the relative frequencies will approximate to the probability: *S1: Whenever we do more repetitions the frequencies will tend to the "central value."*

However, students did not know the nature of this approximation, in particular if there are runs with smaller or greater relative frequencies than the probability, and how long these runs would be. They also did not know what the speed of convergence to the "central value" would be.

Mediational Trajectory

The "box model" applet interacts with each element of the epistemic configurations, and hence it changes the meaning of mathematical objects (Tauber, 2001). Indeed, the simulator makes the study of a new situation (comparing relative frequencies and probabilities when the number of experiments increases) possible; the computing and graphing capacities allow students to make many simulations in a short time, which is impossible without a similar tool. The simulator introduces new language, such as *sequences of events*, and *sequences of bar chart representing the frequency distribution*s; as well as some expressions specific to the simulator: the *clicking on the bar chart* to show the values of relative frequencies; *clicking on balls* to define the urn composition, …).

The simulator serves to illustrate the concept of an “unlimited sequence of frequency distributions,” which are displayed by bar charts, and to visually explore the properties of relative frequency sequences in a large number of experiments. Regarding the proof of emergent properties, the simulator also might lead to students’ greater understanding of the behaviour of random phenomena in an empirical and visual, (not deductive), way.

As noted by Heid (2005, p. 348), “Those who have studied the use of technology in mathematics teaching and learning have noted that technology mediates learning. That is, learning is different in the presence of technology.” This interaction between the mediational and students’ cognitive trajectories was shown in our experiment in three different ways: a) The software visually might show some “stability and convergence of relative frequencies” toward theoretical probabilities in a small sample (e.g., n=37); which might reinforce students’ belief in the false “law of small numbers”; b) The screen graphical resolution interfered with the visual perception of the deviations between frequencies and probabilities; and c) The slow speed of drawings from the urn makes the simulation of very large samples difficult.

Consequently, the influence of technology in teaching and learning mathematics depends of the type of resources available and how they are used. In our case, the simulator did not allow students to follow up the stochastic process generated. Relative frequencies of heads and tails change with every new extraction; but only “accumulated value” of these frequencies are displayed, with no records of previous values. An important consequence is that the “history” of the process is lost.

In Figure 1b we display a screen of Stamedia, another applet that solves this problem, since in this case the differences between relative frequencies and probability (final deviations) are numerically displayed. It is also easy to observe the existence of large runs where $f_r < p$ (or $f_r > p$), and that changes in sign of the deviation are “slow” (Feller, 1989).

Cognitive Configurations

From video and audio recording of interactions between the pair of students, the lecturer and the simulator we gained partial access to the subjects’ personal knowledge regarding the stochastic phenomena studied. We thus observed both incorrect intuitions and probabilistic biases, both well identified in the research literature on the theme, as well as the lecturer’s reactions to them.

Gambler fallacy: In the following excerpt we can clearly observe the “gambler fallacy” or belief that even in small sequences of experiments, the relative frequencies of the event should balance.

L: What do you think about the results coming out?
A2: The tendency should be 50%, the probability should balanced.
L: What do you mean?
A2: That if two 1 come out then two 0 should follow so that they get equal numbers.
L: Does it happen, what you expected?
A2: Of course, it is as if in a roulette game we bet red and it comes out black, you bet again double amount to red and at the end you finish rewarded.

Incorrect intuition of limiting behaviour in stochastic processes: Students click “*Show theoretical probability*” and a line representing the probability appears on the screen.

A1: This is the central value that we should get, as more repetitions happen …
A2: It is 0.5.
A2: As long as we make more draws we will be closer, it isn’t?
A1: I guess it will tend to the central value as we produce more repetitions.

Belief in the “law of small numbers” (The experiment is stopped after 54 tossing when students observe that the frequencies obtained are similar):

L: Do you remember when it was centred in the case of coins?
A1: I think it was about 50 repetitions.
A2: No, it was at 38 draws.

In the second simulation, the frequencies obtained after 50 draws were 0.36 and 0.64, Students were "astonished" at this result but they did not decide to increase the number of draws.

SUITABILITY OF THE STUDY PROCESS

We distinguish three dimensions in the didactical suitability of a teaching and learning process: (Godino, Contreras and Font, in press): a) Epistemic suitability or adaptation between the implemented institutional meaning and the reference intended and global meanings; b) Cognitive suitability or adequacy of implemented meaning to students' potential zone of development: and c) Instructional suitability: possibility that the didactical configurations and trajectories for the lecturer and students identify and solve semiotic conflicts. This also includes the adaptation of technical media, in particular the study time.

Regarding epistemic suitability, as we previously commented the intended meaning (partially fixed in the practice notebook) did not include situations where analysing large sequences of relative frequencies was necessary, which is a key feature of the empirical law of large numbers reference meaning. Therefore observing the probabilistic convergence of relative frequencies to probability was difficult.

Cognitive suitability was low, due to deficiencies in the simulation tool (slow speed to simulate large series of experiments) and in the lecturer's interventions. Certainly, the intended "content" is highly complex if we take into account the students' initial knowledge. It was doubtful that students were able to "reconstruct" the law of large number and its features without adequate guidance by the lecturer. However, had the tasks been better planned and students received a good explanation by the lecturer at key moments of the study process, they would have better understood the intended knowledge.

The interaction pattern between students, lecturer and media could be viewed as "dialogical," so that at a first glance it would have been possible to identify the semiotic conflicts. But due to the "didactic contract" (it was expected that students carry out a practical, autonomous and constructivist work), and the lecturer's lack of awareness of the intended knowledge complexity, and probabilistic reasoning biases in his students, the opportunities to overcome the conflicts were lost. On the other hand, the time to study the two simulator modules and the probabilistic knowledge involved was very short. Moreover, as commented before, the simulator did not allow recording of the "history" of the deviations between frequencies and probability.

A didactical phenomena identified in these interactions was the "teacher passiveness": in some critical moments of the study process where semiotic conflicts arose, the teacher was unaware of these conflicts and he did not introduce the necessary changes in the epistemic and mediational trajectories, thus not taking advantage of the context created to support his explanations.

SYNTHESIS AND IMPLICATONS

In summary, the main conclusions of the study are as follows:

1. The written materials given to students, (that reflect the intended meaning of the study process) were partial and very sketchy, and did not include questions leading the students to observe the behaviour of large series of simulations, runs and the speed of convergence. This is a serious shortcoming from the point of view of the "empirical law of large numbers reference meaning" that makes it difficult for students to observe the behaviour of deviations of relative frequencies and probability, and the occurrence of runs.
2. The time assigned to explore the applet and study the probabilistic knowledge involved was too short.
3. Students were unable to find a probabilistic explanation for the simulation results and stochastic convergence. They remained at a purely perceptive and empirical level, without understanding the nature and justification of the regularities or irregularities they observed in the sequence of events and relative frequencies.
4. The lecturer assumed a "constructivist attitude," expecting students themselves were able to construct the intended knowledge. He neither intervened in the cognitive conflicts

manifested, which were clear as regards stochastic convergence of relative frequencies to probability, nor provided information to solve such conflicts and improve learning.

5. Regulation (institutionalisation) moments, where the lecturer's explanations are potentially effective and necessary, might happen at any point in the didactical trajectory. However, a deep knowledge of statistical objects reference meaning is required from the lecturer, who should also master the specific didactical knowledge about the content to be taught, in this case, knowledge about the main biases and incorrect intuitions on the behaviour of random sequences.
6. It would be necessary to complement the "box model" applet with other resources that allow representation of relative frequency sequences to help solve the cognitive conflicts observed. It would also be necessary to study the a priori probabilities of events linked to random experiments in which the relative frequencies converge, using a traditional technology of "pen and pencil."

The data collected and our reflections show the complexity of the lecturer's work, who should have deep knowledge of the reference knowledge, students' cognitive configurations regarding the intended mathematical content, be able to identify semiotic conflicts and make decisions at critical moments about suitable actions. The management of technological resources in each moment of the didactical trajectory is also difficult.

The teacher's didactical knowledge should include the epistemic dimension (institutional meaning, their adaptations and chronogenesis), the cognitive dimension (personal meanings, cognitive conflicts describes in the literature), the instructional dimension (pattern of interaction, types of didactical configurations, their articulation, optimisation of temporal and technological resources) and the affective dimension.

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