USING TECHNOLOGY TO SUPPORT DIAGRAMMATIC REASONING ABOUT CENTER AND VARIATION

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We conducted two design experiments aimed at engaging sixth graders (11 years old) in statistical reasoning about center and variation. We examine in particular students' informal notion of a "modal clump." Using Peirce's concept of diagrammatic reasoning, we analyze the interplay of 1) making plots with TinkerPlots *– a computer data analysis tool, 2) experimenting with those plots, and 3) developing a language to talk about features of the data sets as represented in the plots by reflecting on judgments. More generally, we draw on Brandom's recent work in philosophy to argue that an "inferential" view should be privileged over a "referential" view of teaching and learning statistics.*

PRIVILEGING INFERENCE OVER REFERENCE

The central thesis of this paper is that the meanings of statistical concepts such as mean and variation should be understood in their role in a reasoning practice and that this epistemological view has implications for the pedagogy of statistics. We start with an important point established in Brandom's (2000) philosophy: the inextricable connection between reference and inference. The gist of Brandom's argument is that there can be no reference (e.g., to a concept or a data set) without inference (a reasoning process). Brandom takes issue with the Cartesian paradigm of representationalism that has prioritized reference over inference in the order of semantic explanation. His major work, *Making it Explicit* (1994), makes a powerful case for the reversal of this order—for the prioritization of inference over reference.

It is never straightforward to formulate educational consequences from philosophical positions. However, there is little doubt that common-sense epistemologies of a Cartesian character underpin much educational practice. If it is indeed the case that traditional pedagogy privileges reference over inference, as in the Cartesian paradigm, then the argument made by Brandom concerning the nature of what a concept is has educational significance, if only as a critique of the underpinnings of those traditional approaches. Commonly, pedagogic practice replicates the Cartesian approach placing emphasis on acquiring the meaning of concepts by virtue of their reference without due regard to the inferential space of reasons intrinsic to the very possibility of reference in the first place. For shorthand we will distinguish a referential view from an inferential view in pedagogic practice and sketch different approaches based on those views.

We will characterize a *referential* view as focusing on concepts and graphs as representations or mirrors of some reality – whether physical or ideal. An *inferential* view sees grasping a concept or understanding a graph as mastering the use of the word or graph in a process of reasoning. Knowing, in the inferential view, is seen as participation in a social practice of giving and asking for reasons, and committing to the inferences that are implicit in making those claims. Participation does not require an immediate and full grasp of the explicit meaning of reasons and claims but rather, the ability to inhabit the space in which they operate. Understanding thus emerges in use.

In relation to the reference-inference distinction, Brandom also contrasts atomistic and holistic approaches. In an *atomistic* view, individual concepts can be defined and understood independently of other concepts. The *holistic* view adopts the position that "one cannot have *any* concepts unless one has *many* concepts. For the content of each concept is articulated by its inferential relations to other concepts. Concepts, then, must come in packages." (Brandom, 2000, p. 15-16, emphasis in original) And: "Cognitively, grasp of just one concept is the sound of one hand clapping" (p. 49).

We also contrast pedagogic approaches that seem to be based on referential and inferential views. The *referential approach* is a "topic-topic-topic approach," in which, for example, mean, median, mode, histograms, and box plots are taught atomistically in separate sections. It is assumed that once students have learned the definitions and procedures they will be able to solve a statistical problem by "applying" those. It is further (perhaps implicitly) assumed that definitions represent the statistical objects and that graphs represent data and therefore mirror a situation in the outside world. In this referential view it is generally not taken as important that these definitions refer to statistical objects within a social practice of people who already know about statistics. The *inferential approach* acknowledges that students with their teachers have to take part in the social practice of reasoning. As Brandom's (2000) argues, for any remark "to have conceptual content is just for it to play a role in the inferential game of making claims and giving and asking for reasons. To grasp or understand such a concept is to have practical mastery over the inferences it is involved in—to know, in the practical sense of being able to distinguish, what follows from the applicability of a concept, and what it follows from" (p. 48). The inferential approach therefore focuses on engaging students in reasoning.

From this perspective, we see our goal as instructional designers as providing environments in which students can reason with the concepts and tools they have at hand and that are meaningful to them. It should be noted here that when we refer to *reason* we do not intend to restrict our meaning to thought entailing explicit inferences but want to embrace thought in the broadest sense. [To use a concept entails participating in "the game of giving and asking of reasons." Users of a concept may be more or less actively aware of the reasons underpinning their concept use, but this does not detract from the presence and role of such reasons in their activity (Derry, 2004).] For example, when comparing two distributions, students should experience a need for concepts that are useful for arriving at and supporting a conclusion about whether they differ. In some situations, the mean could be useful provided that the variation of the two distributions is not too large and that the distributions are not too skewed, and so on. Because students are unlikely to know these concepts, we have to start from their own language and judgments but then find ways to help them develop and refine their reasoning.

Privileging an inferential approach over a referential one does not diminish the importance of the role of reference and representation in learning and teaching. On the contrary, the inferential game involves articulating reasons (Brandom, 2000) and making things explicit (Brandom, 1994) within a socio-cultural practice, and this includes paying careful attention to what the students and teacher represent and refer to. In fact, it is in this reasoning process that concepts and graphs gain their referential dimension: Brandom (2000, p. 157) characterizes the emergence of reference as a "social route from reasoning to representing." As is also known from semiotics, links between signs (words, graphs) and objects (referents) are not pre-given—they are established.

Brandom's philosophy provides an epistemology of concepts but does not discuss diagrams or educational issues. For this reason, we turn to the notion of "diagrammatic reasoning" (Peirce, 1976), which proved useful to analyze how students in grades 7 and 8 formed concepts such as spread and distribution (Bakker, 2004; Bakker and Hoffmann, 2005). Diagrammatic reasoning consists of three steps (not necessarily in this order): making a diagram, experimenting with it, and reflecting on the results. In the reflection phase, characteristics of diagrams are described (predicated), and these predicates can become topics of interest in their own right. In this way, students moved from "the dots are spread out" to "the spread is large" and came to use the notion of "bump" to compare distributions as manipulable objects. In this paper, we are particularly interested in students' notion of "clump" as a measure of center and variation, and we use Brandom's theory as a framework for the analysis of students' reasoning.

TWO DESIGN EXPERIMENTS WITH TINKERPLOTS

To substantiate the epistemological framework set out above, the remainder of the paper is devoted to examples from two very similar design experiments in two grade 6 classrooms in the Midwest of the USA. The experiments were part of the revision of the Mathematics in Context curriculum for middle schools (Bakker and Frederickson, 2005) and involved a collaboration with the *TinkerPlots* design project (Konold, 1998). Design experiments are characterized by design cycles of preparing, designing, testing and revision of, in this case, instructional materials (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003). In Class A, 12 of the 22 students had been classified as “English as a second language” (7), “extra reading help” (5), or “at risk” (3), some of them in multiple categories. In Class B, only one student had English as a second language. All students had studied mean, median, and mode in grade 4 or 5. The experiment in Class A lasted for 20 periods of 42 minutes each, and the one in Class B for 18 periods—over a two-week period. As a researcher, the first author provided most of the instructional materials, did observations and interviews, and occasionally co-taught in the classrooms. The third author designed the original fish activity and was present from the 14th period onwards. The classroom sessions and interviews were video- and audiotaped, and we saved all of the student work.

In the first period, students collected data about students’ foot length and made one large “graph-feet-ee” (pronounced as “graffiti”) on the wall in the corridor. Girls’ feet were represented by yellow cut-out feet and boys’ by blue ones. The main question they were investiagting was whether there was a gender difference in foot size. In the first discussion, students tended to look only at the mode or the highest and lowest values in the graph. A few of the students in the Class A tried to calculate the mean but did not succeed. Some of them were able to find the “median foot” in the graph, but they could not explain whether the result of this procedure was Harry, his foot, or the value 23 cm. In referential terms, these students had limited knowledge of what the mean and median were and, in inferential terms, they did not know how to reason with them either. The teacher and first author felt they needed to take a different, holistic approach, and tried to engage these students in statistical reasoning as opposed to a referential and atomistic approach, which the students seemed to be used to.

From the second period onward, students used a computer tool for data analysis, *TinkerPlots* (Konold and Miller, 2005), which is designed particularly for middle school students. In this software, a data set is first shown as an unorganized set of case icons. Using basic operations such as separate, stack, and order, students organize the data into plots such as a bar graph or a dot plot. Once the case icons have been organized along an axis, it is possible to make more-advanced plots such as histograms or box plots. With this “plot construction tool,” students can explore data sets with multiple representations—both conventional and unconventional. The idea behind the basic operations is that they are meaningful to students and offer ample space for discussion and reflection: why did you stack, what does this plot tell you, which plot do you prefer to answer that question? Another advantage is that any plot can be transformed into any other plot by using only a few basic operations, which ideally helps students make connections among different graph types.

REASONING ABOUT SAMPLES ‘GROWN’ BY HAND

According to the story used for the activity in the 14th period, a fish farmer growing genetically engineered (GE) fish claimed that these grow bigger than normal fish. One year after releasing a bunch of normal fingerlings and a smaller number of GE fingerlings into a pond, students were allowed to catch some fish to check his claim. Each student simulated “catching” about four fish from the pond by drawing slips of paper from a box. Normal fish were represented by yellow slips, whereas the paper for GE fish had a slightly lighter color. The length of each fish was written on the slip of paper. Each student had an activity sheet with axes on which they plotted their own data and the data that other students read aloud.

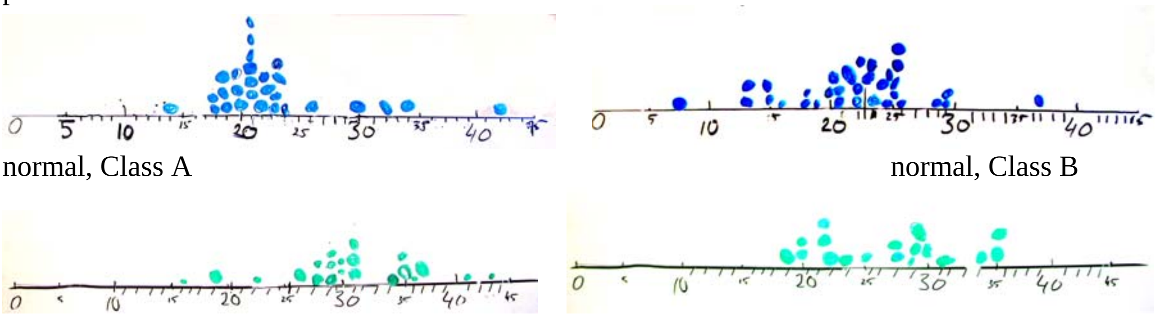

Figure 1: Samples of normal and GE fish on the whiteboard in Classes A and B

Students took turns going up to the whiteboard and plotting their own data. After a couple of turns, students collectively had a closer look at the intermediate result. With regard to Figure 1 (Class A), we first asked for observations to involve them in making judgments. Many of them used the notion of "clump" to refer to the majority of values in the middle of the graph (cf. Konold *et al*., 2002). They had first employed this term in a previous activity that had involved comparing two sets of data. We examine below how the notion of a clump acquired its meaning in a reasoning process and then later functioned as a precursor to a more conventional measure of center (e.g., mean).

As part of engaging students in a reasoning process, we often had to ask where exactly they saw the clumps in the two distributions of Figure 1. Note that the action of specifying this location also addresses the referential dimension of the concept. Students would then answer, for example: "The clump [of the normal fish] is inbetween 18 and 23 [cm]." One boy said that the normal fish were more "spread out" than the GE fish, but also noticed that there were fewer GE fish. In the excerpt below, Linda used the clump notion to indicate both center and spread as she compared the two types of fish while taking the sample size into account (we have more examples of similar remarks).

Linda: Right here [*pointing to the normal fish data*] there is a bigger clump than this one [GE], but the numbers are less. This one [normal], the clump is more spread out but higher.

Res.: You said a *lot* of things. So one was ….

Linda: This [normal] clump is bigger [more values] but has a smaller number [lower values]. And this one [GE] is smaller [fewer values], but it has a higher number [higher values].

We see these as examples of students developing a shared language with which they could articulate different types of variation: between the distributions (GE fish are longer and perhaps less variable), and in the sample sizes (the GE fish samples are smaller, which also causes the lower height of the GE clump). Thus in being pushed to support their contention that the GE fish were longer than the normal fish, they needed to clarify what they meant. As part of this inferential process we had to keep asking them to clarify what they were referring to—the referential dimension: where do you see the clump, what do you mean by "smaller," "higher," or "more"?

The objective of the following episode was to help students connect their reasoning about center and variation to data sets as distributions. We asked students to predict what would happen if they made the sample bigger or if "the sample grew." The examples are from Class B (see Figure 1). Students made several conjectures about what would happen with larger samples. Barbara expected a GE clump to form between 20 and 35 cm, and the different clump sizes seemed to function as ways of informally measuring variation within data sets (e.g., 10-30 vs. 20-35 cm). Anissa said, "The top [normal graph] is a little more spread out, but that may be because there are more." One student thought that the clump would stack up; another thought that the clump would spread out. Sheila said, "I think it will stay in the same area." This is one of the insights we aimed at: coming to see stable features in variable processes. In this example, Sheila expected stability in the area where new data values would appear.

We also asked students to draw a sketch of what they would expect. Some (but not all) students had reasonable intuitions about the shape of the distribution—a concept that is hardly addressed at middle school level, but which can help to bring coherence in reasoning with a number of key statistical ideas. For example, Norman compared the shape of the predicted graph with more data points to that of fathers' heights (roughly a normal distribution), an example he knew from an earlier investigation. After discussing the shape, we returned to the original question of whether the GE fish tended to be longer than the normal fish. Anissa: "I would say that there is more of a variety in the normal fish, but I would actually say that the genetic fish do grow taller, because more of them are closer to the right side.… So I think [the fish farmer] is right." The researcher went on to ask:

Res: How could we find out how big the difference is? If there is a difference….

Norman: What we have to do is go on to *TinkerPlots*, put all the information on *TinkerPlots*, then make a graph and find the mean value, and use a reference line, and find out how far both mean values are apart, if they are apart, and then it'll probably, however far they are apart, that's how much bigger they are.

From such episodes we concluded that their experimenting with the software options helped students to develop a language, anchored in software objects such as reference lines and mean values, with which they could reason about the problems. This does not imply that Norman, for example, knew precisely what mean values were (in a referential sense). But he did seem able to reason (inferentially) with the concept and the tool, and satisfactorily solve the problem without being trained in a particular standard procedure. Of course, we could have probed him further about his notion of the mean (e.g., "why is it OK to use the mean value?"), but at this stage we were satisfied with his answer and were concerned that he was a bit ahead of the rest of the group.

REASONING ABOUT SAMPLES USING TINKERPLOTS

For the next activity, we explored the same context but used a different data set, available in *TinkerPlots*. The question was whether the fish farmer was right in claiming that GE fish were not only longer than normal, but were about *twice* as long. Building on their prior reasoning with clumps, the center clump became a tool for reasoning about aggregates for some students, here exemplified with examples from Class B. Kerry, for instance, compared the two clumps of normal and GE fish as follows: "28–35 and 42–49, which is not twice as much." Moreover, students started to react more to one another than just to the teacher.

Karen: If you say "clump," do you mean where most of the fish are? Or do [*interrupted*].

Kerry: Yeah, like where most of the fish are normal; their length is in between the range 28–35, where most of them are.

This interaction among peers suggests a progression in their engagement with the ideas, that they are taking on the value of asking for reasons and of making claims and ideas explicit.

Our final example illustrates how students' diagrammatic reasoning, supported by the software tool, led to comparing distributions with the mean while taking into account the types of variation between the two data sets. To make the graph shown in Figure 2, Tom separated the fish types vertically, ordered the lengths horizontally, stacked the dots, and used the mean button and reference lines to compare the types of fish. He explained further:

I clicked the mean value and the reference line, because it shows kind of where the clump is. And that helps me because it is easier for me to see where most of them are. And this one [the normal fish] there is a lot more, there is 292 [using the count option]. And this one, there is 67, so there is about a fourth, a little under a fourth. This one [GE] is a lot more gradual, it is spread out, but they grow a lot bigger and this one [normal] is very steep [points with the mouse along the slope] and then is really steep, going down. And you can see that it is not really twice. These ones aren't twice the size of these ones. It is more like one and a half times.

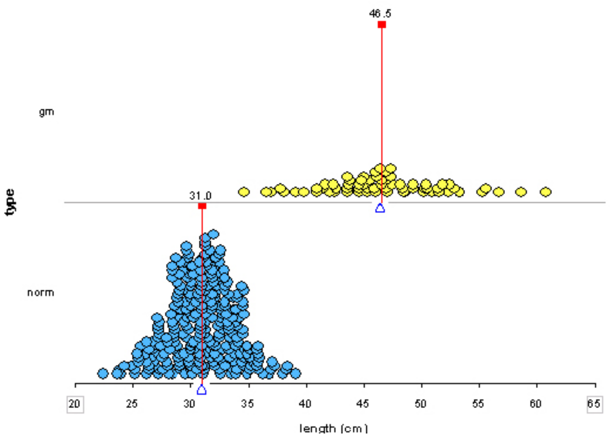

Figure 2: Tom's solution. The blue triangles give the position of the mean values, and the movable vertical red lines are called reference lines

It is interesting that Tom first used the means to indicate where the clumps were, which is the progession we had hoped for: first let students reason with a notion of center (for instance, with clumps), and then let them measure it more formally with mean or median. In that way, the mean is used as a group descriptor while taking into account the different types of variation. In fact, one of the results of the experiments was that we gained a better insight in those types of variation ("variation" is not just one concept!): variation between and within data sets as well as variation in sample size and variation around an ideal shape. In our view, one cannot expect students to use the mean (for example) when comparing data sets without engaging them in reasoning about these types of variation.

DISCUSSION

The examples in this paper illustrate a progression in diagrammatic reasoning—first about "clumps" when arguing whether the GE fish were longer than the normal fish, and later using *TinkerPlots* options and informal terms of describing center and types of variation. Making a diagram (or diagrams) and then experimenting with it are the first and second steps of diagrammatic reasoning—and *TinkerPlots* seemed to support those steps well. In our classroom experiments, the third step—reflecting on the results—seemed best accomplished in small and large group discussions. It was typically the teacher who nudged students to be more explicit, to characterize what they were referring to, and to support their claims with reasons. It was as part of this process of diagrammatic reasoning, which was mostly taking place in the social context of giving and asking for reasons, that statistical features such as clumps, means and variations became topics of attention and "objects to reason with." This exemplifies the "social route from reasoning to representing," as Brandom (2000) calls it. Of course, students' concepts were not yet as precisely defined as they are in statistical discourses, nor were the graphs as sophisticated. But at they were beginning to use their ideas as statisticians do, to make and support claims about distributions. We believe that this provides a fruitful foundation for further development.

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