

STUDENTS' UNDERSTANDING OF VARIABILITY IN A PROBABILITY ENVIRONMENT

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The concept of statistical variation in a probability distribution is closely connected to the concept of sample space in a probability task. One must have some sense of the possible outcomes in a probability task in order to predict the likely variation range that will occur during repeated trials of that probability task. A survey version of a NAEP probability task was given to 652 mathematics students in grades 6 - 12 to obtain information on students' understanding of the sample space. Subsequently 28 students from grades 8-12 were given an interview version that included a simulation of the task. Survey results indicate that a higher percentage of students taking advanced mathematics correctly answered the probability task than was predicted by the NAEP data. Results of the interviews suggest that students who at first thought incorrectly about the probability task were likely to change their minds after seeing the variation in results of sets of repeated trials of the task.

Several years ago in a review and reflection on the literature the first author noted a strange gap in research on student learning in probability and statistics (Shaughnessy, 1997). Much of the research (and teaching for that matter) on students' understanding of statistical concepts was focusing primarily on students' conceptions of the mean, or more generally, on measures of centers in distributions. Many of the tasks used in research ask students to give a single number for an answer, "What is the probability that...?" "How many would you expect...?" Such research questions echo the overemphasis in our mathematics curricula on single outcome predictions, probability of one event, or measures of center, as opposed to spread. One of the recommendations from that review was that research should pay more attention to students' understandings of variation in distributions.

Since that review some work in understanding students' conceptions of variation has begun (Torok & Watson, 2000; Reading & Shaughnessy, 2000; Shaughnessy, Watson, Moritz & Reading, 1999). Much of the initial work has focused on students' understanding of variation in a sampling environment. Results of this previous work provide some support for our conjecture that the teaching of statistics tends to emphasize single outcome responses and to focus on centers rather than spreads, on measures of central tendency rather than on conceptions of variability.

The work presented in this paper had a threefold purpose. First, to extend our investigation of students' understanding of variation to contexts other than a sampling environment, we wished to investigate the connection between students' understanding of a probability sample space and their conceptions of variability. Second, we wanted to gather data on a NAEP (National Assessment of Educational Progress) spinner probability task from students who were actually taking mathematics, and who were from a variety of levels in their mathematics. Third, we wanted to investigate a sample of students' thinking through interviews while the students gathered actual data on the spinner task, to see if the variation in results of trials influenced their decision-making.

THE SURVEY TASK

The original spinner task from the 1996 NAEP is posed below in Figure 1. In the 1996 NAEP study this task was administered to a sample of over 1100 12th graders students drawn from randomly selected classes in schools which themselves were randomly selected from the pool of NAEP volunteer schools. (Schools can choose whether or not they want to participate in the pool of potential NAEP schools.)

A detailed discussion of the NAEP results on this task can be found in previous work (Shaughnessy & Zawojewski, 1999; Zawojewski & Shaughnessy, 2000). To be receive the top rubric score the NAEP students had to disagree with Jeff (there is not a 50-50 chance) and also provide correct reasoning, i.e. list the sample space (BW, BB, WB, WW), or use the

multiplication principle ($1/2 \times 1/2 = 1/4$), or sketch an area model demonstrating that there was only a 25% chance of winning the spinner game. Only 8% of the 12th graders in the NAEP study provided both a correct answer and correct reasoning for the spinner task. An additional 20% of the NAEP students said that there was not a 50% chance of winning the game, it had to be less than 50%. Many students in this category reasoned “if there were one spinner, the chance of winning it would be 50%, so it has to be less with two spinners.”

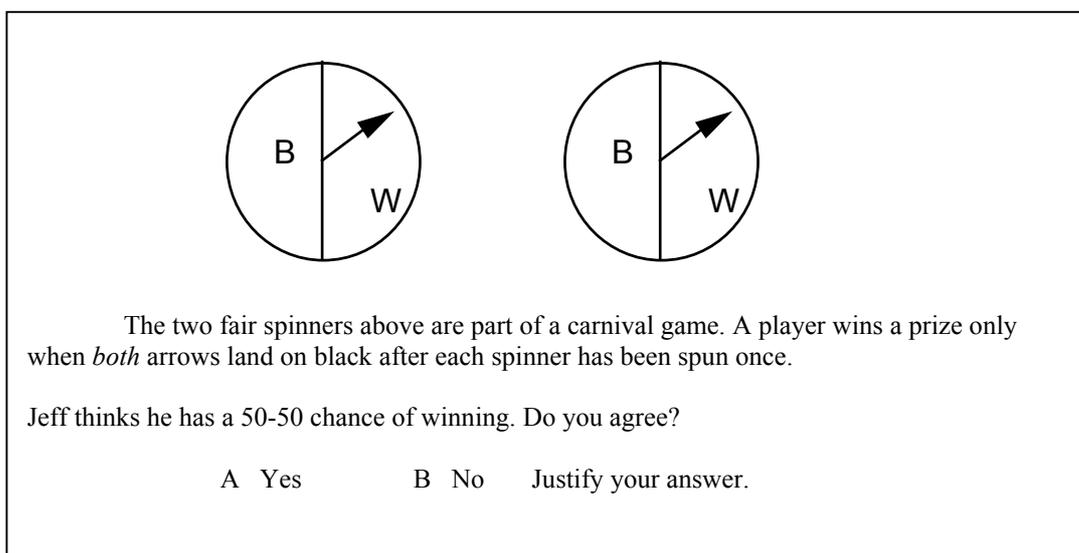


Figure 1. Spinner task from the 1996 National Assessment of Educational Progress (NAEP).

The NAEP results were a bit discouraging to us, as such a small percentage of 12th grade students answered correctly. On the other hand, most of the 12th graders in the NAEP study were not taking mathematics in grade 12, and might not have had any mathematics for several years. Furthermore, the NAEP test is low stakes for these students, as it has nothing to do with their grades. We wished to find out what students in actual mathematics classes, across a range of grade levels, would do with this task. We also wanted to pose the task in a more statistical way than its original probabilistic form. We wanted students to imagine a series of games, to predict the percentage of wins and losses, and then to actually gather data from playing a series of games themselves. Would seeing the variation in results of multiple plays affect their answers?

SUBJECTS AND PROCEDURES

During the 1999-2000 school year the survey form of the spinner task was administered to 652 students, grades 6 – 12, in 28 mathematics classes in 5 schools. Three schools were suburban, one urban, one rural. The teachers of the mathematics classes of these students (both the survey and interview students) had been participants in a professional development project on the teaching and learning of statistics the summer prior to the school year when the surveys were administered. The surveys were administered to the students by the researchers during the regular mathematics time, with the assistance of the classroom teachers. The spinner task was one of four statistical tasks on which students were surveyed. One of the other tasks explored students’ conceptions of variability in a sampling environment.

RESULTS AND DISCUSSION OF THE SURVEY TASK

A summary of the results of the survey is given below in Table 1. Classes from grades 6-9 tended to be grade level specific, while classes at the upper grade levels were normally mixed grades 10-12, or 11-12 inclusive. In each grade level column (6, 7, etc.) in Table 1 the Y – N label is followed by response frequencies for classes in that grade level. The labels “Y – N” indicate the number of students who agreed that yes (Y) the spinner game was 50 – 50, or no (N) the game was not 50 – 50.

Table 1
Summary Results of Student Responses to the Survey Version of Spinner Task

Course				<i>Alg, IMP</i>	<i>Alg</i>	<i>Pre-calc</i>	<i>APStat</i> ³	<i>APCalc</i> ³
Grades	6	7	8	9, 8-11	10-12	10-12	11-12	11-12
	Y - N	Y - N	Y - N	Y - N	Y - N	Y - N	Y - N	Y - N
Results	13 - 4	21 - 3	17 - 3	18 - 11	13 - 7	0 - 16	2 - 15	1 - 37
for	12 - 4	34 - 9	17 - 6	11 - 8 ²	11 - 4	3 - 17	4 - 27	
each	16 - 3	14 - 4	15 - 9	26 - 17 ²	11 - 8		3 - 23	
separate								
class	16 - 2		25 - 6		12 - 4			
			16 - 4 ¹		18 - 7			
					15 - 9			
					12 - 8			
Totals	57 - 13	69 - 16	90 - 28	54 - 36	92 - 47	3 - 33	9 - 65	1 - 37
N								
%correct	19%	19%	24%	43%	34%	92%	88%	97%
or partly								
correct								

Notes: 1. This class was an 8th grade algebra class.
 2. These classes used the IMP, Interactive Mathematics Curricula.
 3. AP refers to Advanced Placement, secondary courses that can lead to college level credit.

The percentages of correct answers (“No, Jeff does not have a 50-50 chance of winning this game”) for each grade/class level given in Figure 1 include: students who gave correct answers with correct reasoning (i.e., listed sample space; used multiplication principle); students who gave correct answers with partially correct reasoning (“It must be less than 1/2 chance of winning as there are two spinners”; “The chance is 1/3 as there are three outcomes, BB, WW, BW”); and students who either guessed or had some non statistical reason for their correct answer (“You can never tell with spinners”; “Carnival games are rigged”; “Anything that you believe can happen”). These latter non-statistical responses were rare, and occurred mainly among the younger students in our sample (grade 6’s and 7’s). In all classes, at all levels, the predominant reasoning of students who thought there was a 50-50 chance of winning the game was based on the half-black, half-white areas in the spinners. (“They’re half of a circle, so 1/2 is 50% and the other half is 50%”; “Both spinners have an equal amount of B and W so it’s 50-50”; “The circles are exactly in half”).

Only about 20% of the grade 6-8 students in our sample felt the chances of winning the game were less than 1/2. There is some growth indicated in the grade 9 algebra students and the IMP classes up to around 43% correct, but then there is a decline among grade 10-12 algebra students to the mid 30% range for percentage correct. This isn’t surprising, as those students who delay taking algebra to grades 10-12 are likely to be among the more mathematically challenged students, whereas those in the IMP or 9th grade algebra are more mathematically able (grade 9 algebra) or more likely see some probability (IMP students). Only at the upper grade levels, in the advanced mathematics courses (pre calculus, AP stats, AP calculus) do a high percentage (90% or more) of the students answer the spinner task correctly. Additionally, almost all the students in the advanced mathematics classes who answered the question correctly also supplied correct reasoning for their answer. Overall, the data for our students in grades 9-11 who were enrolled in introductory level secondary mathematics courses gave responses similar to the NAEP sample of grade 12 students (in which 28% gave the correct answer; and only 8% gave the correct reason). On the other hand, the students in our sample who continued to take 3 or 4 years of secondary level mathematics, those who were enrolled in advanced coursework, did very well on the spinner task.

As we suspected, grade 12 students who were actually enrolled in mathematics courses did much better on the spinner task (90% compared to 28%) than the general population of

grade12 students from the NAEP study. The students in these advanced mathematics courses are clearly familiar with simple compound probability problems such as the spinner task. They have probably encountered similar problems in either their pre calculus coursework, or most certainly in their AP statistics classes. Unfortunately, our data indicate that contrary to what is recommended by the NCTM (2000) *Principles and Standards for School Mathematics* document, our students in grades 6-11 enrolled in mathematics coursework at the algebra level or below did not appear to have much experience with probability. In particular, a high percentage of these students were unable to list the sample space for the spinner task. If students can't list the sample space, there is little chance that they will have any feeling for the variability that could occur in repeated samples of plays of the spinner task. Perhaps playing the actual game, and seeing the variability, can help students to recognize the sample space for the spinner task. To investigate this we devised an interview script for the spinner task.

THE INTERVIEW FORM OF THE TASK, SUBJECTS, AND PROCEDURES

Interviews on several the spinner tasks were conducted with 28 students in grades 8 – 12 from urban and suburban high schools located in a large metropolitan area in the Northwest part of the United States (1 Gr 8, 14 Gr 9, 3 Gr 10, 9 Gr 11, and 1 Gr 12 student). The students were taking mathematics classes that ranged from pre algebra through pre calculus, and were from different schools than the students who took the survey form of the spinner task. However, the schools of the interview students were located in the same urban and suburban regions as the schools of the survey students. Thus, the students in both the survey and interview schools had a very similar demographic profile. The interviews took about 25-30 minutes, and were administered during regular mathematics time to students in a quiet location in a separate room in the school. The interviews were audio taped (we did not have permission to video these students). Students were first given the spinner task in the same way it was asked in the surveys, and asked to give a reason for their answer. The rest of the interview was as follows.

a) If we did 10 trials of the game, how many would you expect would be winners? _____
(Student and researcher then play the game 10 times with the actual spinners and students record the data anyway they like.)

b) Now that we have played the game, what do you think? Do you agree with Jeff that there is a 50 – 50 chance of winning this game? Explain why or why not?

c) What would you expect to happen if we played the game 10 more times? _____
(Students are then given the opportunity to gather more data by playing the game more times with the researcher).

d) What is your final prediction? Is there a 50 – 50 chance of winning? Why or why not?

RESULTS AND DISCUSSION OF THE INTERVIEWS

The results of students' responses to the interviews are summarized in Table 2. Prior to gathering data, 13 students felt the spinner game was not fair, but only five reasoned correctly that the chance of winning was 1/4. Four of these five listed the sample space while 1 used the multiplication principal. The other students who initially felt the game was unfair said it had to be 'less than 50%', because one spinner would make the game 50% wins. Thus, prior to gathering data, about 18% of the interviewees initially gave both correct answers and correct reasons for why the spinner game was unfair.

Table 2
Summary Frequencies for Students Interviewed on the Spinner Game

N = 28	<i>Before Data</i>	<i>After Data</i>
<i>Yes, Game 50-50</i>	15	5
<i>No, Game not 50-50</i>	13	23
<i>Correct reasons</i>	5	12
<i>Incorrect reasons</i>	23	16

After gathering data for several sets of 10 plays of the spinner game, 4 of the 5 students who initially claimed the game was unfair maintained their position and their reasoning and an additional 11 were heavily influenced by the data and changed their minds to “No, it’s not 50-50.” Eight of these 11 who changed their minds actually listed out the sample space *after* playing the game, something they didn’t think to do during their initial response to the spinner game. The remaining students who said the game was not 50-50 either had maintained, or changed to, reasoning that “it must be less”, but these students didn’t correctly list the sample space. (The one student who was correct at first, and changed his mind to reasoning incorrectly did so after becoming very confused while working on another spinner task that was a later part of the interview protocol).

CONCLUSIONS

Seeing the actual data, *seeing the variation in repeated samples of ten trials*, led a number of our students to construct the sample space for the spinner problem. Prior to gathering data, only 4 of the 28 students listed the sample space. After gathering the data, 12 students listed the sample space, unprompted by the researchers. Also, after generating their data more students realized that the game *had to be less than 50%* for wins. The results of the interviews lend some support to our thinking that there is a connection between the concept of sample space in probability and the concept of expected variation of the value of a random variable in statistics. Knowing the probability distribution, the sample space, can inform students when they predict a likely range for the number of wins in ten plays of the spinner game. Conversely, information from gathering samples of ten plays of the game, the *range* of the random variable (X = number of wins in ten plays), can lead students to actually discover the sample space on their own. The conceptual root of the pedagogical power that we gain from having students conduct simulations is the connection that they can make between the observed variation in data in repeated trials of an experiment, and the outcomes that they expect based on a knowledge of the underlying sample space or probability distribution.

There are also some pedagogical concerns raised by our research. The five students who still felt the spinner game was 50-50 after gathering the data maintained their beliefs despite clear empirical evidence to the contrary. The data these students obtained in their sets of ten plays ranged from 0 Wins, 10 Losses up to 3 Wins, 7 Losses. These students seemed to completely ignore the results of their trials. Perhaps as one of them said, “*Anything can happen when you play, the game should be 50-50*” best describes their belief system. For these students, there doesn’t seem to be a connection between the data, and the structure of the game. This particular result is just one more instance in a long line of research results that demonstrate how difficult it is to change some students’ beliefs about probability and statistics (Shaughnessy, 1992). The other concern raised by this research is that there is strong evidence that the students in the survey portion of this study had not received adequate instruction about simple probability problems. This spinner task is a very elementary compound probability problem, and overall our students were very weak on it. We clearly have a long way to go to implement the recommendations for probability made by the Principals and Standards documents (NCTM, 2000).

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