FRAMEWORK FOR TEACHER KNOWLEDGE AND UNDERSTANDING ABOUT PROBABILITY ®

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Theoretical frameworks for analyzing teacher subject-matter knowledge in specific mathematical domains are rare. In this paper we propose a theoretical framework for teacher subject-matter knowledge and understanding about probability. The framework comprises of seven aspects: essential features, the strength of probability, different representations and models, alternative ways of approaching, basic repertoire, different forms of knowledge and understanding, and knowledge about mathematics. We explain the importance of each aspect for teacher knowledge of probability, discuss its possible nature and illustrate our claims with specific examples.

INTRODUCTION

The teacher's role includes setting mathematical goals and creating classroom environments in which these goals are pursued, classrooms in which all students encounter, develop, and use mathematical ideas and skills in the context of genuine problems and situations, and where the teacher chooses appropriate ways to represent the subject matter, asks questions, suggests activities and guides discussions. To do this, teachers need adequate knowledge. But what is adequate knowledge for teaching mathematics in the way described above?

Our research centers on the knowledge teachers need in order to teach probability. Shulman’s (1986) three categories of content knowledge in teaching: a) subject matter knowledge, b) pedagogical content knowledge, and c) curricular knowledge, serve as a basis for our theoretical framework. Our research includes two interrelated components: theoretical and empirical. In an interconnected process we construct a theoretical framework for teacher professional knowledge of probability and examine this knowledge in actual classroom probability teaching. In this paper we present only one element of our work: a theoretical framework for Shulman’s first category of subject matter knowledge. We chose to focus on this category because, while no-one would argue with the statement that teachers need to have adequate subject matter knowledge, what "adequate" might mean is rarely discussed. Consequently, we do not discuss here other important aspects related to the teaching of probability. The framework for teacher subject-matter knowledge and understanding about probability presented here is an extension and elaboration of Even’s (1990) general framework of subject matter knowledge for teaching a specific topic in mathematics and its application to the case of function.

The construction of the framework for teacher subject-matter knowledge of probability is grounded in: (a) examination of the role and importance of probability in mathematics as well as in other disciplines and real-world situations, (b) the role of probability in the mathematics school curriculum, (c) research and theoretical work on learning, knowledge and understanding of mathematical topics and concepts in general and probability in particular, and (d) research and theoretical work on teachers’ subject matter knowledge and its role in teaching. We examined mathematics books on probability, synthesized published research on understanding and learning of probability, reviewed relevant textbooks and other curriculum materials, and interviewed mathematicians and mathematics educators. We aimed at crystallizing *what* is it that they consider to be important about probability and how this may be related to teacher subject matter knowledge about probability. Critical evaluation of the information obtained from these various sources served as a basis for the construction of the framework presented in this paper. In the following we present the framework’s seven aspects of teacher knowledge and understanding about probability, explain their importance, discuss their possible nature and illustrate our claims with examples. The seven aspects represent an interconnected whole.

THE FRAMEWORK: ESSENTIAL FEATURES

One aspect of the framework deals with the essence of probability, as it is important that teachers know the essential features of probability, what makes it different from other fields in mathematics:

- Probability is the mathematical way to deal with problems of *uncertainty*. It is a tool for measuring the appearance chance of events. In this regard, probabilistic thinking is fundamentally different from deterministic thinking that is used in other mathematical topics that generally relate to the present or the past.
- Two approaches exist for handling questions about probability: The *Objective Approach* and the *Subjective Approach*. The two approaches differ from each other by the meaning attached to the term Probability.

The Objective Approach. This is the most common approach and it governs most probability textbooks. According to this approach, one cannot assign probability to a one-time event. Probability can be assigned only to an event that can be repeated, like tossing a coin or drawing cards from a pack. Hence, the claim that "the chance of drawing a Jack is 4/52" is interpreted as a claim about the relative occurrence of Jacks in repetitive drawings, under the same conditions. This approach diverts the discussion of uncertainty from a single occurrence (such as the drawing of a single card) to a series of occurrences (like a sequence of card draws). However, in many cases we want to relate to one-time events that cannot be repeated in exactly the same way. For such cases we need to switch to a subjective approach to probability.

The Subjective Approach interprets the term probability as the degree of belief rather than as a relative occurrence. If we interpret probability as the measure of a confidence level, we can then apply the term probability to one-time events, such as assessment whether an alleged murderer is innocent, calculation of the chance for a successful operation for a specific patient, or estimation of the chance to win an election. This approach is called "subjective" because according to this approach probability represents a subjective judgement made by the individual and not an objectively measurable characteristic. According to this approach, different people may allocate different (subjective) probabilities to the same event (such as election results), if they have different information or scope of view. However, for events such as tossing a coin or drawing a card, we would expect that the subjective approach coincide with the respective relative frequency. The subjective approach treats probability as a language for describing the level of uncertainty that one feels. As any language, it has its own syntax, that is the rules of probability calculation (Liberman & Tversky, 1996). It is important that teachers be aware of the two different ways to approach the meaning of probability.

- The term probability describes the extent of the *predictor's knowledge* and not the event. For example, conditional probability describes an update of the predictor's knowledge about a particular event when additional information is provided.

THE STRENGTH OF PROBABILITY

The strength of probability, what makes it important and powerful, is also an important aspect of teacher subject matter knowledge. Probability opened new possibilities for dealing with uncertain and random situations that occur in almost every field of our life. Today, probability has become an integral component in every area. Already three decades ago, Freudenthal (1970) remarked that "Probability applies in everyday situations, in games, in data processing, in insurance, in economics, in natural sciences. There is no part of mathematics that is as universally applied, except, of course, elementary arithmetic" (p. 167).

Falk and Konold (1992) state that they expect that in the 21st century understanding probability will be as important as mastering elementary arithmetic was in the 20th century. As Hacking said, "Today our vision of the world is permeated by probability, while in 1800 it was not. Probability is the great philosophical success story of the period" (Hacking, cited in Kruger, Daston & Heidelberger, 1987, p.45). The term "probabilistic revolution" (Kruger, Daston & Heidelberger, 1987) signals a shift from a deterministic conception of reality, phrased in terms of universal laws of stern necessity, to one in which probabilistic ideas have become central and indispensable. The revolution goes further, however, viewing chance as an integral part of natural phenomena. This conception is represented, for example, in quantum physics. An equally dramatic example of the probabilistic revolution that has changed our thinking about our own existence is found in evolutionary biology.

DIFFERENT REPRESENTATIONS AND MODELS

Work in probability, as with in any other mathematical topic, is conducted via different representations and models, such as, table, Venn Diagram, area model, tree diagram, pipe diagram, formulas, etc. Familiarity with different representations and models and the ability to translate and form linkages among them create insights that allow a better, deeper, more powerful and more complete teacher understanding in probability. For example, the tree diagram is a useful tool for computing probabilities associated with series of events (dependent or independent). In contrast, the area model is limited to events with at most two steps. However, the latter model is convenient for computing conditional probability, as the ratio between areas of rectangles is visual. The pipe diagram (Konold, 1996) provides a concrete representation of the important property that the sum of the probabilities of all mutually exclusive outcomes of a random event is 1. In terms of the pipe metaphor, water from one column or stack is split among the pipes in the next column and the total amount is, therefore, always equal to the original amount. In contrast, the sum of the accompanying conditional probabilities, in all but the first column of the tree diagram, is greater than 1. But, the pipe diagram is more cumbersome as it requires more space than the tree diagram, and requires more time to draw.

ALTERNATIVE WAYS OF APPROACHING

In addition to the use of different representations and models, alternative ways of approaching are also used in complex fields, such as probability. Teachers should be familiar with the main alternative approaches and their uses, and the need to make good choices among different available approaches. There are two main approaches for the definition of the term *probability* in the objective school: *The classic approach* (Laplace) – The probability of an event is the ratio between the numbers of results that fulfill the desired event and the number of elements in the sample space, when the sample space is finite and uniform. *The experimental (frequency) approach* – the probability of an event is the value at which the relative probability stabilizes when the number of experiments is large enough. The alternative approaches are different from one another and none of them is suitable for all situations. For example, when a Doctor says that: "If you take this medication you have 70% probability to get well". This sentence means that a large enough number of experiments were made, in which the relative frequency of recovering patients was observed. This type of sentence cannot be interpreted by the classic approach. On the other hand the following sentence can be interpreted using both approaches: The probability that the outcome of tossing a coin is tail is 0.5.

BASIC REPERTOIRE

Part of teacher knowing and understanding mathematical topics or concepts is to know and have easy access to specific examples, which constitute a basic repertoire. A basic repertoire includes powerful examples that illustrate important ideas, principles, properties, theorems, etc. Some of the examples are simple and illustrate a single aspect. Others are complicated and present several terms and principles simultaneously. A basic repertoire also contains mathematical terms and sub-topics that are connected with probability, like terms from set theory and combinatorics. A basic repertoire for teachers should include representative important examples from the probability curriculum.

To illustrate the idea of basic repertoire, let us look first at two simple examples that can be included in a basic repertoire. *Independence* may be illustrated with the case where a fair cube was rolled twice and resulted in "3" both times. As the rolls are independent—the cube does not "remember" the previous results—the probability that the outcome of the third roll will be "3" as well is 1/6. *Sample space* may be illustrated by the case where "head" and "tail" comprise the *sample space* of a coin tossing.

The following two examples present more complicated principles. The first one "*The Monty Hall Problem*" (appears in Selvin, 1975) shows that new information updates the probability estimate. In a popular television show, the player faces the dilemma of having to choose among three doors—two of which hide a goat behind them and only behind one door there is a car (the prize). After making the choice which door to open, but before opening the door, the master of ceremony (who knows where the car is) opens one of the two other doors where there is a goat behind it. At that point the player can reconsider and change the original choice. Should the player make a different choice now?

Let us mark the three doors by A, B and C, and assume that the player has chosen door A. If the car is behind door B, the master of ceremony opens door C. If the car is behind door C, the master of ceremony opens door B. If the car is behind door A, the host opens (with equal probability) either door B or C.

Because probability describes the state of knowledge of the evaluator and not the event, then, for the contender, when starting the game, the probability that the car is behind door A is 1/3 as is the probability that the car is behind door B (or C). But, in view of the additional information received (by the opening of one of the doors by the host), the probability that the car is behind door A is no longer the same as the probability that the car is behind door B (or behind door C). To illustrate the change in the probabilities, we assume that the host had opened door B (the case of C is similar). Then, the required probability is therefore P(A/b), where P(I) is the probability that the car is behind door I, and P(i) is the probability that the host opens door I, namely, the probability that the car is behind door A when it is known that it is NOT behind door B.

$$P(A/b) = \frac{P(b/A)P(A)}{P(b/C)P(C) + P(b/A)P(A) + P(b/B)P(B)} = \frac{0.5*1/3}{1*1/3 + 0.5*1/3 + 0*1/3} = 1/3$$

Hence, for the contender, the probability that the car is behind door A remains 1/3 while the probability that the car is behind door C is now 2/3 (because the probability that the car is behind door B is now zero). This example illustrates how adding new information updates the probability estimate.

Confusion between P(M/B) and P(B/M) is common. The *"Medical test results"* example (appears in Liberman & Tversky, 1996) could serve to show the significance of distinguishing between the two. A hospital is considering whether to use a new diagnostic kit in order to identify a specific disease. Data gathered over the years show that 15% of the population are infected by this disease. A well-designed experiment conducted in order to evaluate the efficacy of the diagnostic kit shows that 8% of the sample came out "positive" (i.e., sick), out of which 90% were actually sick. In other words, if M designates the group of patients that received positive results and N the group of patients that were actually sick, then P(N/M)=0.9. This result may seem good. However, P(M/N), i.e., the probability that sick people will indeed be identified as such using the diagnostic kit, may be more relevant for the hospital. According to Bayes' theorem

$$P(M/N) = \frac{P(N/M)*P(M)}{P(N)} = \frac{0.9*0.8}{0.15} = 0.48$$

These results may not be satisfactory and the hospital may choose not to use the test.

DIFFERENT FORMS OF KNOWLEDGE AND UNDERSTANDING

Knowledge and understanding are terms with multi-meanings and interpretations. The mathematics education literature describes these terms in various ways. Different researchers mention various forms of knowledge and understanding. These include, for instance, conceptual, procedural, instrumental, relational, formal, algorithmic, intuitive, knowing that, knowing how, knowing why, and knowing to. Some researchers claim that a dichotomy exists between different types of knowledge and understanding and advocate a specific type (e.g., Skemp, 1978). We, like some others (e.g., Fischbein, 1993; Nesher, 1986) claim that knowledge and understanding of specific mathematical topics and concepts appear in different forms, and it is the combination and integration of the different types that is empowering. This is especially true in the case of probability, and consequently, in the case of teacher probability knowledge, where intuitive knowledge often misleads the problem solver. For example, when looking for the probabilities of receiving "head" zero times, once or twice when tossing two coins, the intuitive naive answer is usually that the odds are equal. Using a formula, or analyzing the different cases conceptually, assist in obtaining a correct answer (Fischbein, 1987). When compared with other mathematical fields, the use of intuitive knowledge in probability leads, in many cases, to wrong answers and hence using and integrating other types of knowledge could serve as control.

KNOWLEDGE ABOUT MATHEMATICS

Knowledge about the nature of mathematics is interrelated with probability knowledge. Knowledge about the nature of mathematics is a more general knowledge about the discipline, which guides the construction and use of different types of knowledge. It includes ways, means and processes for the establishment and creation of truths as well as the relative importance of different ideas. The nature of mathematics also includes it being a creation of the human mind, which is influenced by different forces inside and outside mathematics, and the characteristic of the constant change of mathematics. Sometimes, the more general knowledge about mathematics *supports* probability knowledge, other times it *withholds* it, as is illustrated in the following.

I. *Supporting knowledge*

- In an axiomatic system, including in probability, no contradictions are allowed. For instance, values of the probability function are never greater than 1. Hence, it is impossible for the probability of the union of events to be greater than 1. The use of such knowledge can monitor a common oversight of the intersection of events.
- Work with mathematical concepts, which are abstract, requires the use of models, each of which is limited and presents only some facets. Similarly in probability, different models have to be used, according to the situation.
- Inductive and deductive reasoning form the basis of work in mathematics. Experimentation (i.e., the use of particular cases) supports the making of conjectures. However, inductive reasoning is not sufficient. A proof, based on logical reasoning is also required in mathematics. This is also true, of course, in the case of probability where experimentation is a common means for establishing hypotheses.

II. *Withholding knowledge*

- Limits appear in mathematics in different circumstances: limit of a series, limit of a function, etc. Limits appear also in probability when we say, for example, that if we repeat an experiment "a large number of times" the relative frequency will approach the probability of the event. However, unlike the series case, for instance, where the limit involves the existence of a natural number N for every $\varepsilon>0$, so that from that place onward, the distance between each and every member of the series and the limit is smaller than ε, it is not possible in the probability case to find such a number N that corresponds to a specific ε.

Teacher probability knowledge should include knowledge about the nature of mathematics and awareness to its support or withholding of probability knowledge.

CONCLUSION

Today, probability has become an integral part in every field of our life, such as, insurance, economics, medicine, physics, biology. Consequently, a growing number of countries include probability in their school curricula. Still, many teachers avoid the teaching of this subject. Shaughnessy (1992) suggests that one reason for this might be insufficient teacher subject knowledge. Research on probability understanding and its development conducted in the last three decades indicates that children and adults experience difficulties when solving probability problems and learning probabilistic ideas, and that intuitive knowledge of probability often leads to wrong decision making. Teachers have a critical role in supporting student learning. Teachers' ability to fulfill this role is connected to their subject matter knowledge. No one would argue with the claim that teachers need adequate subject-matter knowledge. However, what adequate might mean is not clear. Theoretical analysis of what subject matter knowledge for teaching a specific topic in mathematics might mean is lacking, in mathematics in general and in the case of probability in particular. In this paper we propose a theoretical framework for teacher subject-matter knowledge and understanding about probability. The framework includes seven interrelated aspects, each accompanied by an explanation for choosing it, that together represent a whole. But perhaps as important is what is missing from the framework: It does not include a long list of what teachers need to know, as we do not believe that such an approach to teacher subject matter knowledge is useful. This omission, together with the choice of the aspects, the explanations and reasoning accompanied each choice, and the emphasis on the aspects as representing an interconnected whole, are all part of the framework. This framework serves us in our empirical study on teacher probability knowledge. Moreover, we expect that, in a dialectical process, our empirical research in classrooms will again influence the theoretical frameowrk. The framework can serve researchers who study teacher probability knowledge by clarifying and making explicit what such knowledge might mean, as well as guide the development of courses for pre-service and in-service teachers.

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