DEVELOPING INSTRUCTIONAL CONJECTURES ABOUT HOW TO SUPPORT STUDENTS' UNDERSTANDING OF THE ARITHMETIC MEAN AS A RATIO ®

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This paper develops insights into how to deal with the arithmetic mean in the initial phases of statistics instruction. It focuses on one of the various ways in which the mean can be used in statistics: as a normalized ratio. The paper analyzes individual interviews of 12 12-year-old students prior to their participation in a classroom teaching experiment. These interviews were used to test and develop instructional conjectures about how to support students' understanding of the mean and other normalized ratios. Three differences where detected in how students seemed to make sense of proportional comparison problems in which the use of the mean as a ratio is pertinent. The paper explains how these findings can be capitalized upon in designing and conducting instruction.

BACKGROUND

The need to develop a "pedagogical definition" of the arithmetic mean that can be used as a guide for instruction, and for assessing students' thinking and learning, has been a recurrent challenge in the statistics education literature. Mokros and Russell (1995) identified the model of dealing with the mean as "the fair share" as the most commonly used for introducing the concept. These authors considered this model inadequate for guiding instruction. Instead, they proposed that the central ideas should be determining what is "typical" or "representative" in a data set.

Thompson (Cortina, Saldanha, & Thompson, 1999) found Mokros and Russell's pedagogical model unsatisfactory from a quantitative-reasoning perspective (Thompson, 1994), and defined the mean as "a ratio that measures group performance relative to the number of contributors in the group" (Cortina, Saldanha, & Thompson, 1999, p. 466). Konold and Pollatsek (2001) developed yet another alternative, which consists of dealing with the arithmetic mean –and other measures of central tendency– through the general orientation of what is signal versus what is noise in a data set.

The recurrent concern of developing a suitable pedagogical definition of the arithmetic mean reflects the diverse ways in which this construct is used in statistics. In a sense, the arithmetic mean has diverse "mathematical personalities," some of which can be associated with different statistical practices that involve the use of this construct (cf. Cortina, 2001a). For example, approaching the mean as a center can be associated with practices that involve the analysis of normally distributed data sets, and notions of "*tendency"* (as in "central tendency") can be associated with practices that focus on carrying out probabilistic inferences.

The mean can also be associated with statistical practices in which issues concerning distribution or probability are not central. In this paper I focus on one of such practices: That in which the mean is used as a ratio that allows comparisons of total accumulations across groups or entities that are different in size (Cortina, Saldanha, & Thompson, 1999). My goal is to offer insight into two important pedagogical issues related to this way of using the arithmetic mean: first, the relevance, for statistics instruction, of students coming to understand the arithmetic mean in this way (i.e., as a ratio). And second, instructional conjectures about how to support students' understanding of the mean as a ratio.

MEAN AS A RATIO

The purpose of this section is to offer the reader a general idea of what is meant by "the mean as a ratio," and to explain the significance of students coming to understand this way of using the arithmetic mean. Apparently, the first uses of the mean to emerge in the history of mathematics were related to dealing with issues involving "total accumulations" (c.f. Bakker, 2000), one of which was estimation. For example, an ancient Hindu problem (quoted by Bakker) was based on estimating the total number of leaves and fruits in a tree by multiplying the approximated mean number of leaves and fruits in a branch by the total number of branches in the tree.

Another way in which the mean deals with totals is in its use as a normalized ratio that addresses the multiplicative relation between the total accumulation of a certain property and the number of contributing units that generated it (Cortina, Saldanha, & Thompson, 1999). For example, a mean national income (or income per capita) can be thought of as the multiplicative relation between the national total income and the number of people that contributed to that total. These kinds of multiplicative constructions are useful for comparing aggregate qualities across groups or entities that are different in size. For example, if two countries had equal population size, it would be possible to assess which was the wealthier country by comparing their income sums. If the populations were not equal in size, it would still be possible to carry out "wealthiness" comparisons by using normalized ratios that addressed the proportional relation between the nations' total incomes and the size of their populations (i.e., their income per capita).

A normalize ratio is a mathematical tool that can serve to assess proportional relations in a standardized way (Freudenthal, 1983). For example, ratios can be constructed with the total accumulation of distances covered by cars and the amounts of gasoline that it took them to do so (e.g., Car A, "400 km/40 L", and Car B, "600 km/50 L"). These "original ratios" can be rescaled into a standardized unit (e.g., "400 km/40 L = 10 km/1 L"; and "600 km/50 L = 12 km/ 1 L"). The rescaled ratios can be the basis for carrying out comparisons of qualities that are considered to be dependent on the proportional relation between the two quantities that form the ratio. For instance, "kilometers per liter" can be used as a measure of the quality "fuel efficiency" that is considered to be dependent on the proportional relation between the distance that a car covers and the volume of gasoline it requires to do so. "Income per capita" can be a measure of the quality "national wealthiness" that is dependent on the proportional relation between a nation's total income and its population.

The arithmetic mean can be thought of as a special case of a normalized ratio; that in which one of the original quantities that constitute the ratio is a total accumulation of a property (i.e., the sum of values), and the other the number of discreet amounts that contributed to that total (i.e., "N"). The two examples I have used fit into this definition; in the case of income per capita, "total income" can be thought of as the sum of personal incomes (i.e., sum of values), and "population size" as the number of personal incomes that were involved in the sum (i.e., "N"). In the case of "kilometers per liter", total distance can be consider as the sum of distances that a car covered as it sequentially consumed "N" liters of fuel. There are other normalized ratios that are not arithmetic means. For example, "books per pupil" (in school libraries) is not an arithmetic mean because the total number of books in a library cannot be said to be the sum of the books possessed by individual pupils.

One thing to notice when the mean is used as a normalized ratio is that variability across the size of the individual accumulations that create the sum is overlooked. In the examples I have given, this means that variation on individuals' income, or on the distances covered by a car as each liter was consumed, need not be acknowledged. Instead, the attention focuses on the proportional relation between two quantities (i.e., the sum of values and "N"), which is considered to measure a phenomenological quality (e.g., "the wealthiness of a country" or "fuel efficiency of a car").

The use of the arithmetic mean as a normalized ratio is not, in a strict sense, statistical because it does not deal with issues related to statistical variability or prediction (cf. Cobb, McClain, & Gravemeijer, in press). However, in the work that my colleagues and I have conducted exploring ways of supporting middle school students' learning of statistics, we found normalized ratios (most of which were arithmetic means) to play an important role in certain kinds of statistical analyses.

A central feature of the instructional framework we use when conducting classroom teaching experiments is to have students deal with realistic problems (c.f. Cobb, McClain, & Gravemeijer, in press; Cobb, Stephan, McClain, & Gravemeijer, 2001; Tzou, 2000). While developing instructional activities on statistics, we found that units of measure that are normalized ratios are used in many situations in which scientists and social scientists conduct statistical analysis, specially analyses involving attention to statistical covariation. This indicated that in order for students to develop abilities to conduct the kinds of statistical analysis of bivariate data that are common in the sciences and in social sciences, they need to be able to make sense of arithmetic means and other normalized ratios that are used as units of measure (e.g., “parts per million”, “income per capita”, etc.). Thus, it seems sensible to include normalize ratios as part of statistics instruction –even if they are not used as *strictly* statistical constructs in the sense already elaborated.

CONJECTURES ON HOW TO SUPPORT STUDENTS' LEARNING

In this part of the paper I present an analysis of research findings on students’ ways of interpreting and reasoning about proportional comparison problems. The purpose of this analysis is to develop instructional conjectures on how to support students’ learning about the arithmetic mean and other normalized ratios.

The data comes form individual interviews conducted in the spring of 2000 with twelve 7th-grade students (twelve-year-olds), prior to their participation in a classroom teaching experiment on normalized ratios. The methodology used for designing and analyzing the interviews was based on the first phase of the Classroom-Teaching-Experiment research methodology as developed by Cobb and colleagues (cf. Cobb, 1999; Cobb, McClain, & Gravemeijer, in press; Cobb, Stephan, McClain, & Gravemeijer, 2001). Consistent with this methodology, prior to designing and conducting the interviews (and the teaching experiment), conjectures were developed about: a) what might be the big mathematical ideas involved in coming to make sense of normalized ratios when used as units of measure, b) a possible learning path that could guide students to become able of making sense of normalized ratios, and c) possible starting points for instruction (cf. Cortina, 2001b).

The basic instructional conjecture that emerged was that it might be possible to help students make sense of normalized ratios by building on the underlying mathematical notions that would come into play in their thinking when dealing with proportional comparisons, in situations that involved unequal size groups. The interviews were conducted with the intention of exploring the viability of this conjecture, and of anticipating starting points for instruction. Specifically, the interviews were expected to fulfill two goals: 1) to determine if students would find it reasonable to interpret the instructional activities that we[1] were planning to use in the teaching experiment as involving proportional comparisons; and 2) given that these interpretations occurred, to document the mathematical notions that would emerge in students’ reasoning, in relation to how these emergent notions could inform instructional planning.

The participating students were attending an urban middle school in the southeastern United Sates. Two researchers were present during the interviews; one asked students questions while the other took field notes. During the interviews, students were presented with three problems.[2] Students were asked to justify their answers, and to explain their reasoning as they interpreted and tried to solve the problems. The researchers asked probing questions whenever they felt a need for clarification. The interviews lasted from thirty to forty-five minutes and were video recorded.

One of the problems presented to students asked them to compare the sales of Girl Scout cookies in different years. Students were told that in the year 1998 a 10-member Girl Scout troop had sold 778 boxes of cookies, and that in 1999 the troop had grown to 15 members and had sold 1002 boxes. Students were asked to determine, on the basis of the information for 1998, whether the troop should be satisfied with the number of boxes sold in 1999. As a follow-up question, students were told that in the year 2000 the troop was expecting to sell 2000 boxes. The number of troop members for that year was 25. Students were asked to determine if the goal set by the troop was reasonable, based on the information from the previous years. A second problem that students considered consisted of comparing earnings in tips between two restaurants (see Figure 1).

These two problems shared common characteristics that we hoped would make them reasonable for students to interpret as proportional comparisons: a) both problems entailed comparing qualities (i.e., “troop performance” and “restaurant profitability”) that could be considered to be dependent on the proportional relation between two quantities (e.g., total boxes sold and number of troop members); and b) both presented information only about totals.[3]

Martha is thinking of working as a waitress during weekends. There are job openings in two restaurants; wages in both are the same, but she does not know how much she can expect to earn in tips. Laura, who works on the weekends at The Pork Belly Grill, told her that she earns a different amount in tips every night, but that she has been saving all her tip money for the past three months (12 weekends) which amounts to a total of $ 2040. Another friend, Monica, who works at the Metro Café, says that in five months (20 weekends) of waiting tables there, she has made $ 3200 in tips.

Laura	The Pork Belly Grill	12 weekends	$ 2040
Monica	Metro Café	20 weekends	$ 3200

[How do you think Martha could use this information to decide where to work?]

Figure 1. Restaurant Problem

FINDINGS

Nine of the twelve students who participated in the interviews interpreted at least one of the two problems as involving a proportional comparison. This confirmed the potential usefulness of these kinds of instructional activities, since for most of the students it seemed to make sense to engage in solving the problems in ways that were pedagogically productive (i.e., by caring out proportional comparisons). In the analysis of the interviews, special attention was given to differences in students' reasoning that could become the focus of instruction. Three significant differences were detected. I will discuss them in the order in which they might become relevant in an instructional sequence.

The first difference relates to ways in which students' interpreted the problems. As I have all ready mentioned, not all students seemed to find it reasonable to interpret the problems as involving proportional comparisons. This meant that, in designing instruction on supporting students' learning of normalized rations, an initial instructional goal should be to assure that all students come to understand "the reasonableness" of dealing with the problems by carrying out proportional comparisons.

A second significant difference in students' ways of dealing with the problems relates to how students interpreted and manipulated quantities when carrying out the proportional comparisons. In most cases, students' proportional comparisons were based on making the two ratios "the same size" by "growing" one of them. For example, in order to compare troop performance between 1998 (10 members) and 1999 (15 members), students estimated how many boxes would the troop have sold in 1998 had it had 15 members. To accomplish this, students "copartitioned" the two original accumulations of 1998 (see below). The ways in which these copartitions were constructed and manipulated seemed to indicate that there were two significantly different ways in which students were making sense of them.

In some cases, students seemed to treat the copartitions that they constructed as multiplicands that were useful for estimating total accumulations in relation to different group sizes. For example, in order to determine how many boxes of cookies 15 troopers would have sold in 1998, the original quantities of that year would be copartitioned (e.g., "778 boxes ÷ 10 = 77.8 boxes", and "10 troopers ÷ 10 = 1 trooper"). The resulting quotients (e.g., "77.8 boxes" and "1 trooper") would then be multiplied to estimate accumulations for different troop sizes (e.g., "77.8 boxes" × 15 = "1167 boxes" for 15 troopers).

In other cases, students seemed to treat the copartitions as "the missing pieces" for completing certain accumulations. For example, to determine how much the troop would have sold in 1998, had it had 15 members, students would determine the corresponding sales of boxes for 5 troopers, which would give them "the missing piece" (e.g., "778 boxes" ÷ 2 = "389 boxes" and "10 troopers" ÷ 2 = "5 troopers"; "778 boxes for 10 troopers" + "389 boxes for 5 troopers" = "1167 boxes for 15 troopers"). Both the "multiplicand approach" and the "missing piece approach" were sensible and efficient ways of dealing with the problems. However, the "multiplicand approach" seemed closer to the notion of a normalized ratio (e.g., "for every one member in the troop, 77.8 boxes were sold").

This difference in students' activity was considered instructionally significant because it could be capitalized upon in advancing a mathematical agenda (McClain & Cobb, in press). If a teacher were to be able to make the difference in how students interpreted and used copartitions "visible" to her classroom, productive conversations could be held about how copartitions can be used and understood when dealing with proportional comparisons. These conversations could help everyone in the classroom come to make sense of how, why, and when, a copartition can be used as a multiplicand. This, in turn, could become an important step in helping students make sense of the mathematics behind normalized ratios.

The third significant difference in students' ways of dealing with the problems was related to the final outcome on which students based their comparisons. I have already explained how, in most cases, students conducted the proportional comparisons by "growing" one of the ratios and making it "the same size" as the other. There were also four students who –at least on one of the problems– made use of normalized-ratio-like constructs. For example, in comparing troop performance between 1998 and 1999, the two ratios were copartitioned into "N=1" (i.e., "778 boxes ÷ 10 = 77.8 boxes", and "10 troopers ÷ 10 = 1 trooper"; and "1002 boxes ÷ 15 = 66.8 boxes" and "15 troopers ÷ 15 = 1 trooper"). The comparisons were then based on the "per one units" (i.e., "in 1998 the troop sold 77.8 boxes per trooper, and in 1999, 66.8 boxes per trooper"). The difference between the two kinds of outcomes on which to base proportional comparisons indicated that, at a certain point in instruction, it might be possible to conduct whole-class conversations that focussed on the difference between using a partition as a multiplicand to make a ratio grow (i.e., using it as a tool), and using it as a measure in its own right (i.e., a normalized ratio). These conversations, by being grounded on students' activity and contributions, could become effective means by which to further advance the mathematical agenda.

In general, findings from the interviews suggest that our basic instructional conjecture was viable. That is, it seems possible to help students make sense of normalized ratios by building on the underlying mathematical notions that would come into play in their thinking, when dealing with proportional comparisons in situations that involved unequal size groups. Moreover, the interviews offered valuable information about differences in students' reasoning that could be the basis for addressing significant mathematical issues during instruction.

FINAL REMARKS

Incorporating the diverse "mathematical personalities" of the arithmetic mean into the K-12 curriculum in a pedagogically coherent way has been a recurrent challenge for statistic educators. In this paper I have proposed to look into uses of this construct that can be considered "pre" or "nonstatistical," but nevertheless significant for conducting statistical analyses. I have explained the pedagogical importance of helping students come to make sense of arithmetic means and other normalized ratios. I also hope to have contributed to the understanding of students' intuitive ways of dealing with proportional comparisons in situations in which the use of means as normalized ratios are pertinent, and possible ways to capitalize on the diversity of those intuitions in instruction.

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NOTES

[1]Paul Cobb, Kay McClain and my self conducted the teaching experiment.

[2]Findings on only two of these problems are discussed in this paper.

[3]A pilot study showed that students did not seem to find it reasonable to carry out whole group proportional comparisons when information about individual data was given (e.g. information about how much each trooper had sold, or about how much was earned each weekend; c.f. Cortina, 2000b).

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