TEACHING INTRODUCTORY RANDOM WALK

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Simple random walk is taught at the beginning of a course on stochastic processes. Most of the topics in random walk are quite new to the students. The topics have to be taught with as many links to elementary probability as possible. This paper outlines an approach together with some discussion of the difficulties that students encounter.

INTRODUCTION

In theory, by the time students begin to study simple random walk, they would have acquired some degree of sophistication in mathematics including probability theory. Unfortunately this is not always the case. It is found to be necessary to include a lot of explanations and reminders of probability theory and other mathematical results or topics as one teaches simple random walk. Examples are given and then tutorial questions that are similar to these examples are then set. Quite often students are still unable to do tutorial problems that are similar to class examples. This paper attempts to present and address aspects of simple random walk that the students find difficult.

POSITIONS OF RANDOM WALK

It seems easier to begin the discussion of simple random walk by considering the random movement of a particle on the X-axis. The particle is assumed to perform a simple random walk on the set of integers. Let the particle be initially at 0. At time n, the particle takes one positive step to the right, i.e. moving say from i to i+1 with probability p and one negative step to the left with probability q =1-p where 0 < p < 1. Let $X_n,\ n = 0,1,2,...$ denote the position of the particle at time n. Let $Z_i,\ i = 1,2,3,...$ be independently and identically distributed Bernoulli random variables with $P[Z_i = 1] = p$ and $P[Z_i = -1] = q,\ i = 1,2,3,...$. Then $X_0 = 0$, and $X_n = Z_1 + Z_2 + ... + Z_n = X_{n-1} + Z_n,\ n = 1,2,...$ The usual graphical representation of a realization of a simple random walk with the X_n displayed on the Y-axis and the time displayed on the X-axis is a useful pictorial representation. This may be supplemented by the use of a tree diagram for a small value of n. As an illustration, the evaluation of some probabilities such as: $P[X_n \geq 0 \text{ for } n = 1,2,3,4]$, and $P[|X_n| < 3 \text{ for } n = 1,2,3,4]$. A tree diagram will show up very clearly all the possible realizations of $\{X_1, X_2, X_3, X_4\}$. As n increases, the tree diagram becomes very large. Students will soon realize that a tree diagram is only an aid to the solution of some problems about X_n. The possible values of $Y_n = X_{2n}$ and $W_n = X_{2n+1}$, $n = 0,1,2,...$ are then discussed. This leads on to the discussion of the evaluation of $P[X_n = k]$. Students appear to need very detail explanation of the derivation of $P[X_n = k] = \binom{n}{n+k/2} p^{(n+k)/2} q^{(n-k)/2}$.

If one then poses the following question: What is the probability that at time n there are k steps to the right? The majority of the students can work out the answer and yet they usually cannot give the value of X_n. Students may also take some time to figure out the answer to the question: What are the values of n and k that will make $P[X_n = k] = 0$?

A discussion of the derivations of $E[X_n] = n(p-q) = n\mu$ and $Var[X_n] = 4npq = n\sigma^2$ then follows. The Central Limit Theorem is then used to obtain approximate answers to such questions as (a) Find $P[X_n \geq j]$ and (b) Find $P[a \leq X_n \leq b]$. At this stage, students may not have been taught the concept of confidence interval yet. So it is not possible to discuss the confidence interval of X_n. However, it may be possible to point out that

$P\left[n\mu - 3\sqrt{n\sigma^2} \leq X_n \leq n\mu + 3\sqrt{n\sigma^2}\right]$ is almost 1. Students have no difficulty in applying the Central Limit Theorem to obtain the relevant approximate values.

THREE LEMMAS

Assuming a general initial position X_0 so that

$X_n = X_0 + Z_1 + Z_2 + ... + Z_n$, $n = 1,2,3,...$

Some properties of the simple random walk are taught in the following three lemmas.

Lemma 1 $P[X_n = j \mid X_0 = a] = P[X_n = j + b \mid X_0 = a + b]$.

Lemma 2 $P[X_n = j \mid X_0 = a] = P[X_{n+m} = j \mid \mathrm{X_m} = a]$.

Lemma 3 $P[X_{n+m} = j \mid X_0, X_1, ..., X_n] = P[X_{n+m} = j \mid \mathrm{X_n}]$.

The distribution of the partial sum of independently and identically distributed random variables is still a very new concept for the students. Some may have difficulties in believing $P\left[\sum_{i=1}^{n} Z_i = j-a\right] = P\left[\sum_{i=m+1}^{m+n} Z_i = j-a\right]$.

More time has to be spent in explaining the proofs of the lemmas especially Lemma 2. No mention of spatial and temporal homogeneity and the Markov property will be made at this stage.

THE GAMBLER'S RUIN

The gambler's ruin problem may be approached by considering the motion of the particle in the interval $[0,c]$ where $c = a+b, a \geq 1, \mathrm{b} \geq 1.$ If the particle starts from the position i, what is the probability that it will reach $0\ (\text{ or } c)$ before it reaches $c\ (\text{ or } 0)$? Let $u_i\ (\text{ or } v_i)$ be the required probability. In solving the problem, initially a "first step analysis" is used. This involves the analysis of the possibilities that can arise after the particle has taken a first step from its present position and then applying the law of total probability. Most students appear to have difficulties in understanding the derivation of the relevant difference equation. It is necessary to go through the concepts of mutually exclusive events and the law of total probability once again before proceeding with the solution. To obtain the solution of u_i one needs to solve a difference equation. Students may not have come across difference equation in other courses yet, so a general solution to difference equation will be given. This is apply to the present problem to obtain:

(a) For $p \neq q,\ u_i = \dfrac{r^c - r^i}{r^c - 1}$ where $r = q/p$.

(b) For $p = q,\ u_i = 1 - (i/c)$.

When $p = q$, the probability of reaching 0 is dependent on the distance from 0. Students should find v_i in the same manner just for the exercise and then show that $u_i + v_i = 1$, $i = 1,2,...,c-1.$ The sum of $u_i + v_i$ also indicates that the particle will almost surely not remain in any finite interval forever.

As an example, introduce the problem in the context of gambler's ruin and discuss the idea of absorbing barrier. A proof that the game is a fair game if and only if $p = q$ is also given.

When $c \to \infty$, $\lim\limits_{c\to\infty} u_i = \begin{cases} r^i & q < p \\ 1 & q \geq p \end{cases}$

This may be explained as the case of a gambler playing against an infinitely rich adversary such as a casino.

CONCLUDING REMARKS

Simple random walk is the first part of a course on stochastic processes. The discussion of topics such as the first passage time, the duration of a game or time to absorption will be taught in connections with topics in Markov chain.