DEVELOPMENT OF THE CONCEPT OF WEIGHTED AVERAGE AMONG HIGH-SCHOOL CHILDREN

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The objective of this study was to see how students' strategies for solving weighted average problems change over their high-school years. The questions were: What knowledge of the weighted average do children have before the introduction to the concept? What strategies they use to solve problems through their high-school years? Do they improve after formal instruction? Do they differ across grade levels? Are errors persistent?. A written test with significant situations which covered various contexts and representations was given to high-school students (N= 598) prior to instruction on the average (8^th^ grade), at the moment of instruction (9^th^ grade) and a year later (10^th^ grade). Analysis of the students' papers looked for stable strategies, modifications from one level to another, and the impact of the different variables put in place.

Students' low performance in the area of the average is well documented. Over the years many studies have highlighted student difficulties with average problems, particularly in the case of weighted average. Mokros and Russell (1995) show that children do have conceptions (or misconceptions) of representativeness, and that they see the average in five principal approaches, as mode, an algorithm, a reasonable value, a midpoint and finally a point of balance. Other studies have focused on the properties of the average, Mevarech (1983) observed that college students with basic statistical education had a tendency to apply the four axioms which constitute an additive group to the computation of means. Looking at the development of children's concepts of the arithmetic average, Strauss and Bichler (1988) observed difficulties among 8-14 year old children in understanding that the sum of the deviations is zero, in taking account of a value of zero in the computation of the average, but particularly in understanding that the average is representative of the values averaged. Leon and Zawojewski (1990) testing similar problems with an older subject group concluded that most people can understand the mean as a computational construct, but have more difficulty understanding it as a representative value.

This situation seems to be true at all ages. Cai (1995) found that although 90 % of sixth–grade students questioned knew the computational algorithm for the average, less than half of them had a conceptual understanding of the concept. This finding was consistent with the earlier results of Pollatsek (1981). Out of 37 college students, only fourteen were able to compute a weighted average problem. Finally, a exploratory study

(Gattuso, Mary, 1996) indicated that although performance in “textbook” problems increases from high-school to university, this is not the case for problems asking for some form of “reversibility”. For example, in a problem asking for a missing value while the final average and the partial average were given, high-school students performed better using more “concrete” strategies like the difference of totals. This implies that they at least implicitly understood that the average is the value that each observation would have if they were equal.

It is clear that the average and particularly the weighted average (Pollatsek, Lima, Well, 1981; Gattuso, Mary, 1995) is not a simple computational algorithm and that it is not well understood. Pollatsek suggested looking at different contexts and presentation formats, while others have experimented using different forms of the balance model (Hardiman, 1984; Mokros, Russell, 1995). From a constructivist point of view, we found it was important to look at the development of high-school children’s concept of the weighted average.

RESEARCH QUESTIONS

This study was designed to investigate the following questions:

- What knowledge of the average have children acquired before their introduction to the concept of the weighted average?
- What strategies are used to solve problems of weighted average by children through their high-school years?
- Do strategies improve after formal instruction? Do they differ across grade levels? Are errors persistent?

The answers to these questions will ultimately guide the construction of future teaching experiments.

METHOD

Subjects

Although simple arithmetic average problems are part of the elementary mathematics curriculum, it is only in 9^{th} grade that Quebec students encounter weighed average in school mathematics. A total of 598 high-school students (age 13-15) participated in the study. There were 241, 8^{th} year students, 239 9^{th} year students (immediately after instruction on the average) and 118, 10th year students (one year after instruction).

Task and administration

Each student answered 5, 6 or 7 questions depending on whether they were in 8th, 9th or 10th grade respectively. A total of 24 different tasks was designed, consisting of 12 different types of problem with variation of context, numbers and structure. Each one of the 24 tasks was administered to at least a third of each year group.

For a closer examination of students' strategies, it was important to present them with significant situations. We produced a bank of items expected to encourage various reasonings and which took account of variables such as context, problem structure, and proprieties of the average.

DESCRIPTION OF TASKS

The tasks can be grouped into four categories. The first category consists of various problems asking students to find a weighted average[1].The second category of problems covers tasks asking the effect of a change in the data on the average .The third category of problems ask for missing data while the partial and total average are given. And the final question is formulated in terms of deviation from the mean. The last category of questions were given only to the older students (10th grade). Only the first category of tasks will be discussed in this paper.

In the first category there are ten different tasks. Three of them presented the data in the form of a table. The context was one of salary (x_i) and number of persons (f_i) and the numbers used varied. The three following tasks were formulated verbally and the context was different for each one. And finally, in the four last tasks, the frequencies were relative and given in terms of percentage or ratio.

The didactic variables

First it was essential to see if the children could find a weighted average, and to evaluate the variables influencing their performance and their strategies. In the first group the problems were presented in a table format, the frequencies were absolute and the context was one of salary frequently encountered. The numbers chosen were simple because calculators were not allowed. We hoped that this would encourage the subjects to pay more attention to their solutions. Since data presented in table format are seen in grade 7, it was expected that 8th-graders would understand the question even though they had no formal instruction on grouped data or weighted averages. The numbers were chosen to counteract errors diagnosed in a previous study (Gattuso, Mary, 1997). For example, $\sum \frac{x_i f_i}{5}$ was greater than the highest observation value. The $\sum \frac{f_i}{5}$ was or greater than the largest observation, or smaller than the smallest or in between. In one version, the values of the observations were much larger than the frequencies (a), in a second, there were about the same (b) in the third, one of the frequencies was much larger than the others (c). We hoped that this would force students to take account of the frequencies.

In the second group of problems we used three contexts, the weight of persons in an elevator (d), marks (f) (see Pollatsek, 1981) and age (e). The weights of each value were far apart so the average was quite different if the weight was not considered. Again, most errors produce unacceptable answers. In the age version, the average is not an integer. For the last tasks where the weight is presented in a relative format as a % (g, i, j) or ratio (h), we used the same contexts as before to allow some comparison. In two of the tasks (i, j), the question asked what would become of the average if the frequencies were inversed (70% and 30%, instead of 30% and 70%) to see if it would help to emphasise the fact that the observations were weighted.

RESULTS

Performance

Performance varied with the variable but not always as predicted. There is improvement with instruction but it does not persist. The 9th graders generally performed better than the 8th graders and 10th graders. This is not really surprising, since they had recently received instruction on the weighted average. One year after instruction, the 10th graders have the lowest rate of correct answers in seven cases out of ten. In fact, the 8th year group performed better than the 10th graders except in the word problem where the context is marks. The 10th graders also had a higher rate of success in the tasks where the context is weight, and the frequencies are presented as percentage. Before concluding, it is important to have a closer look at the students' strategies.

Table 1: % Right Answers

#	CHARACTERISTICS		8th	9th	10th	Total
a	Large difference between x_i (salary) and f_i	Table	51.3	63.3	25.0	50.8
b	Same range of value for x_i (salary) and f_i	Table	63.3	53.8	41.5	55.0
c	One very large frequency	Table	57.3	65.0	45.9	58.3
d	Weight/elevator	Verbal	53.2	70.4	50.0	59.5
e	Age	Verbal	56.0	50.0	48.8	52.3
f	Marks	Verbal	22.5	38.3	55.3	35.1
g	Weight/elevator	%	—	49.4	60.0	52.9
h	Age	Ratio	—	63	52.6	59.6
i	Age	%	—	67.5	60	65
j	Weight/elevator	%	—	55.0	73.7	61.0

Strategies

Strategies for this first group of problems were quite predictable but the importance of some of them vary from one group of subject to the other. First we will describe the principal strategies encountered. We can say that all the strategies that give correct answers follow the formula (F): $\frac{\sum x_i \times f_i}{\sum f_i}$ or one of its variations. In the lower grades, there is often a simplification of the problem. Some subjects list the observations (FL), repeating them as many times as the frequency indicated it. Older subjects translated the frequencies into percentage even if they are given in absolute values, and a few subjects of all ages write the frequencies as fractions.

Others put the relative frequencies in a discrete form, using multiples of 10 or 100 when frequencies were given in percentage, or multiples of 4 or 8 when the frequency was presented as a ratio: "There were three times more retired persons than workers…". In all age groups, we saw a "simplification" of the problem, e.g. translating relative frequencies into absolute frequencies. This strategy accounts for many of the right answers in the last group of tasks, particularly among the 9th graders tackling the Age/ratio problem (h: 56.8%). However, lower performance was mostly explained by other strategies.

The wrong answers gave place to a wider variety of strategies. We grouped as "false formulas" (FF) all the wrong procedures where both the values of the x_i and the frequencies (f_i) were taken into account. For example, the sum of the x values was divided by the sum of the frequencies $\frac{\sum x_i}{\sum f_i}$ or the $\sum\left(\frac{x_i \times f_i}{f_i}\right)$. All these strategies have some similarity with the usual algorithm. Others used 5 or 2 (the number of lines or number of different data (x_i) as a divider with various numerators, for example: $\frac{\sum f_i \times \sum x_i}{5}$.

One of the more frequent "false formula" strategies (FF), $\frac{\sum x_i}{\sum f_i}$, is observed particularly in the answers of the 10th grade students, and it explains part of their poor performance in the Table Format problems (8th: 22.4%; 9th: 14.2%; 10th: 33.0%) and in the Verbal Age problem (e) (8th: 7.3%; 9th: 11.3%; 10th: 17.0%). At this stage, the influence of the choice of numbers cannot be determined. Nevertheless, the fact that this strategy came up with an average smaller than the smallest value, 1$, in the version (c) may have controlled its application (a: 28.64%; b: 18.5%; c: 12.5%) but not in version (a). This cannot be generalised because in the first tasks this strategy gave an answer.

The "simple" arithmetic average (AD) where the weight of the data is ignored is encountered in every problem. It was specially important in the Marks problem where it was applied by half of the subjects. This problem has the poorest performance, 35.1%. The assumption that a larger frequency would force students to take account of the weight of each value was not confirmed, the simple arithmetic average is used as much in that version (a) than the others (b, c). In the Table Format problems, the 10th graders applied the "simple" arithmetic average more often than others. It was the opposite situation in the Verbal problems and in the Weight/Percentage problem, where the 8th and 9th graders computed the average without taking account of the weight of each value. Also in the Weight/Percentage problem (g), for the 9th grade: 23.8%).

One strategy that we call total of data (T), $\sum x_i \times f_i$ leads to a wrong answer for most of the tasks, but a right one when the frequencies are given as percentage (g, i, j). It then becomes equivalent to $\sum x_i \times p_i$. In fact, this seems to explain the better results of the 10th graders in the three tasks where the frequencies are formulated in terms of percentage (T: g 31.7%; i: 45%; j: 47.3%).

DISCUSSION

It is difficult at this stage to draw any general conclusions since only the first category of tasks, asking to compute a weighted average, has been analysed. However, we can see a difference through the age groups. The 8th grade subjects appear to be as good or even better than the 10th graders in all but one of the problems they wrote. The influence of recent instruction probably explains the better performance in 9th grade. Although the strategies are the same in each group their importance varies. The 10th graders seem to have some vague memories of the formula but they master situations with percentages much better. The results for the 8th grade students show that it is possible to find a weighted average even without specific instruction. Finally, the impact of different didactics variables such as numbers, context, on the choice of strategies, has to be investigated further.

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[1] We considered "weighted average" in its simplest form, i.e. where there are observations that have an occurrence greater than one and when there is grouped data (see Pollastek, 1981)