Acceptance Sampling

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Statistics is rapidly gaining a place in the mathematics curriculum, primarily because of its wide applicability. In fact, as has often been noted, with the possible exception of calculus, no other mathematical field has such a wide range of applications. One of the primary applications of statistics in industry is the quality control of manufactured items. Quality control refers not only to the manufacture but also to the inspection of the manufactured product. Items manufactured in large quantities, or lots, are commonly inspected for quality before being given to the buyer. (The buyer may also inspect the product, but we consider only the manufacturer's inspection here.)

Some of the principles of sampling inspection, and some surprising consequences of such sampling, are accessible to beginning students of statistics. It is the purpose of this paper to show how these principles may be presented in the classroom. It is hoped that teachers will use this material as an interesting and useful example of statistical practice.

An example of acceptance sampling: We need the formula for the number of distinct samples of size r that can be chosen from n distinct objects. This number is denoted by the symbol ${}^{n}C_{r}$, and is also called a binomial coefficient. It is often read, "n choose r", and we assume the student knows that ${}^{n}C_{r} = n!/(r!(n-r)!)$.

Suppose we have manufactured a lot of 12 items. We decide to choose two items at random and inspect them. A good analogy here is a carton of a dozen eggs; the buyer randomly selects two to inspect. Let's agree that if either of the eggs if cracked, or otherwise defective, the buyer does not buy the carton of eggs.

We could do one of two things (if the resulting sample has one or more unacceptable items):

(1) replace the unacceptable items in the sample with acceptable items; or
(2) inspect the entire lot and replace any unacceptable items found there with acceptable items.

Note that in action (1) the remainder of the lot, outside the sample, is not inspected. This may appear strange in the egg example, but if the inspection is destructive then there is no other choice.

Let's see what happens under plan (1). Assume that the carton contains k unacceptable items.

In this sampling procedure, the average number of unacceptable items after sampling is:

$$\frac{k\,{}^{12\text{-}k}C_2 + (k\text{-}1)\,{}^{12\text{-}k}C_1\,{}^{k}C_1 + (k\text{-}2)\,{}^{k}C_2}{{}^{12}C_2}.$$

This can easily be simplified to 5k/6. This is less than k, so the sampling scheme results in a prediction in the number of unacceptable items purchased on the average.

These results are easily generalised to samples of size r. If there are k unacceptable items in the lot, we purchase k-i defectives with probability

$${}^{12\text{-}k}C_{r\text{-}i}\,{}^{k}C_i \,/\, {}^{12}C_r .$$

So the average number of unacceptable items purchased is

$$\sum_{i=0}^{r} (k - i)\,\frac{{}^{12\text{-}k}C_{r\text{-}i}\,{}^{k}C_i}{{}^{12}C_r} = \frac{k\,{}^{11}C_r}{{}^{12}C_r}\sum_{i=0}^{r}\frac{{}^{k\text{-}1}C_i\,{}^{12\text{-}k}C_{r\text{-}i}}{{}^{11}C_r}.$$

But the numerator of the summation represents all the possibilities when r items are chosen from 11 items. So the summation is $k\,{}^{11}C_r/{}^{12}C_r = k/12\,(12\text{-}r)$ which is less than k. This is, however, an increasing function of k.

Rectifying the lot: In the previous section we replaced only the unacceptable items in the sample with good items. More spectacular gains can be made if, when we find a critical number of defectives in the sample, we replace *all* the unacceptable items in the entire lot with acceptable items. This is called *rectifying the lot.*

We will continue with the sample above, where the lot consists of 12 items, k of which are unacceptable. Let's agree to rectify the lot if our sample, of size 2, contains one or more defectives. (In practice, the lot is rectified only if the sample contains at least c defectives. Here, c = 1, but often c > 1.)

Now to calculate the average number of unacceptable items, we note that we have k unacceptable items with probability ${}^{12\text{-}k}C_2/{}^{12}C_2$ and 0 unacceptable items with probability $1 - {}^{12\text{-}k}C_2/{}^{12}C_2$. So the average number of unacceptable items is $k\,{}^{12\text{-}k}C_2/{}^{12}C_2$. The average percentage of unacceptable items is sometimes called the Average Outgoing Quality (AOQ). As k increases, (here from 0 to 12), AOQ increases, reaches a maximum, and then decreases. In this case the maximum AOQ occurs from k = 4 and is 14.1%.

Rectifying the entire lot on the basis of a sample then limits the percentage of unacceptable items purchased. This limit is often called the Average Outgoing Quality Limit (AOQL).

It may be helpful to see why this always occurs.

Suppose we have a lot of n items and that we choose a sample of size r. The average outgoing quality, if there are k unacceptable items in the lot, is

$$AOQ(k) = \frac{k}{n}\frac{{}^{n-k}C_r}{{}^{n}C_r}.$$

Since k is discrete (k can only be 0,1,2,...,n), we can find the maximum by dividing a term by its predecessor.

$$\frac{AOQ(k+1)}{AOQ(k)} = \frac{(k+1)\,{}^{n-(k+1)}C_r}{k\,{}^{n-k}C_r}. \quad \text{(The divisors } {}^{n}C_r \text{ cancel.)}$$

$$= \frac{(k+1)(n-k-1)(n-k-2)\ \ldots\ (n-k-r)}{k(n-k)(n-k-1)(n-k-2)\ \ldots\ (n-k-r-1)} = \frac{(k+1)(n-k-r)}{k(n-k)}.$$

The terms increase as long as this ratio is greater than 1, that is as long as $(k+1)(n-k-r) > k(n-k)$ or as long as $k < (n-r)/(r+1)$. Since the maximum is an integer and at k+1, the maximum occurs at $[(n-r)/(r+1)] + 1$. If n = 12 and r = 2, this gives 4.

Large lots - approximations: Acceptance sampling can provide impressive gains with large lots; it may be useful to show a numerical approximation for large lots.

Suppose the lot has k defectives and N, the size of the lot, is large. Also suppose we rectify the lot if a sample of size r has any defectives in it. Then

$$AOQ = \frac{k\,{}^{N-k}C_r}{N\,{}^{N}C_r} = \frac{k}{N}\cdot\frac{(N-k)(N-k-1)(N-k-2)\ \ldots\ (N-k-r+1)}{N(N-1)(N-2)\ \ldots\ (N-r+1)}$$

$$= \frac{k}{N}\left(1-\frac{k}{N}\right)\left(1-\frac{k}{N-1}\right)\left(1-\frac{k}{N-2}\right)\ldots\left(1-\frac{k}{N-r+1}\right).$$

So if $x = k/N$, $AOQ = x(1-x)^r$. This function has its maximum at $x = 1/(r+1)$ and this can be used in a good approximation to the maximum of AOQ.

If $r = 100$, $x = 1/101$ and $AOQ = 1/101\ (1 - 1/101)^{100} = 1/100\ (1 - 1/101)^{101}$ and this is approximately $0.01e^{-1}$ or 0.0037.

So, regardless of the quality of the lot, we buy no more than 0.37% unacceptable items!

References

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