Simulations in the Classroom

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1. Introduction

Students enjoy solving problems by simulation; they invariably consider them games and relate them to their own experiences with activities such as chess, Monopoly or Conway's "Game of Life". They are amused at the uncertainty of the outcome of each trial and the variation between results obtained by themselves and their friends. At the same time they are impressed by the long-run trends and similarities which emerge and the general power of simulation as a technique to demonstrate and analyse situations.

With the development of computers and their ability to handle very large numbers of calculations quickly and cheaply, simulation has become an important method for representing mathematically a wide range of phenomena. The techniques are so simple that, even without a computer, quite complicated situations can be investigated effectively in the classroom.

At Southland Boys' High School simulations are included in the course work for Sixth Form Certificate Applied Mathematics and elsewhere in the curriculum from Form 3 to Form 7, that is, with students aged from 13 to 18 years old. The reasons for their inclusion are:

(i) their wide-ranging applications;
(ii) the simplicity of the methodology;
(iii) the experience they give students in elementary experimental design;
(iv) the opportunity they provide for the study of situations that are not readily explored by other methods.

It is also important that simulations are included which analyse phenomena or deal with problems that can be solved by other methods to give an idea of the reliability of simulation techniques.

2. The duel

An argument is to be decided by target shooting. The opponents take it in turns to shoot at a bottle. The one who makes the first hit is deemed to have won the argument. The contestants are Evariste and Bernard.

Capital letters will denote the scoring of a hit and small letters a miss. Suppose Bernard shoots first, then he will either miss, b, or hit, B. If he misses then Evariste has the chance of a shot and may miss, e, or hit, E. Of course, both could miss and the duel would continue. What is the chance (probability) of Evariste winning? To solve the problem, we need a list of all the possible ways Evariste could win. Bernard has to miss on the first shot before Evariste can hit on his first shot - denoted by bE. However, even if Evariste misses, all is not lost because so might Bernard on his second shot, then Evariste could score a hit; this is denoted by bebE. Mind you, he might miss again, but still all is not lost ... Possible winning outcomes for Evariste are bE, bebE, bebebE, bebebebE and so on. His chance overall of winning is the sum of the probabilities of each of these outcomes.

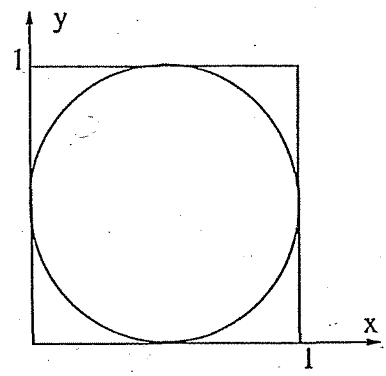
At the moment the problem is unsolvable since there is no way of knowing the probabilities of each opponent scoring a hit. Initially, these could both be set at 1/2 and the duel simulated with a coin - a head indicating a hit and a tail a miss. The coin is tossed alternatively for Bertrand and Evariste and the winner is the one who throws the first head. The simulation is repeated as many times as possible to give the proportion of wins to each duelist.

The problem is also familiar as one involving the sum of an infinite geometric series. The theoretical result can be obtained and compared with that of the simulation to give students an idea of the efficacy of simulation. The comparison also shows that the method of simulation may avoid complicated mathematics.

It is important that students are involved in the experimental design and setting of parameters for the simulation. In the duel students may choose who shoots first and the probability of a hit for each duelist (here, choosing random digits may be used to simulate the shots). Imaginative students also like to change other factors such as what constitutes a hit.

3. Estimating π

This well-known method of estimating the value of π gives students another demonstration of the effectiveness of simulation techniques. Imagine a circle of diameter 1 unit circumscribed by a square of side 1 unit. Since the radius of the circle is 1/2 unit, its area (πr^2) is $\pi/4$ sq. units. The area of the square is 1 sq. unit. If points are chosen at random within the square, a certain proportion of them, namely $\pi/4$, will lie within the circle. If this proportion is known a value of π can be estimated.



Many random points have to be chosen before a reasonable estimate of π can be guaranteed. With 1000 points the maximum error is 0.1. The application is more often used to find areas of shapes which cannot easily be calculated, such as a lake or island on a map.

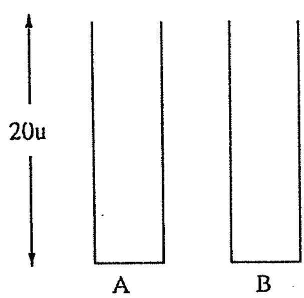
Examples like these give students an outline of the methodology, an idea of its accuracy, and some involvement in experimental design. Further examples need to show the wide scope of simulation techniques, preferably in terms of situations suggested by students.

4. Clustering

A fourth-former was worried - his grandmother was coming to New Zealand for Christmas and there had been a lot of air crashes recently. From this initial comment and lots of discussion involving how to define a coincidence, a simulation was devised to look at how events cluster. The events were chosen as the occurrence of digits in a string of 100 random digits. After voting on a selection of suggestions, a coincidence was defined by the students as the occurrence of two or more identical digits separated by two or less other digits. The degree of coincidence was also defined. Thus the sequence 0 5 3 7 3 0 7 7 2 7 5 4 has 7 in a degree 4 coincidence and 3 in a degree 2 coincidence. By examining lots of strings of 100 random digits the class was able to observe that approximately 44 percent were involved in a coincidence. An unexpectedly high number to the students who had guessed beforehand at a figure of between 10 and 20 percent. One student suggested that maybe coincidences weren't!

5. A packing problem

This next problem also arose out of class discussion and created a great deal of interest among students.



Two bins, A and B, each of capacity 20 units are to be loaded with items. Each item is of integral size from one to nine units. The items arrive singly at random and each is loaded before the next is sighted.

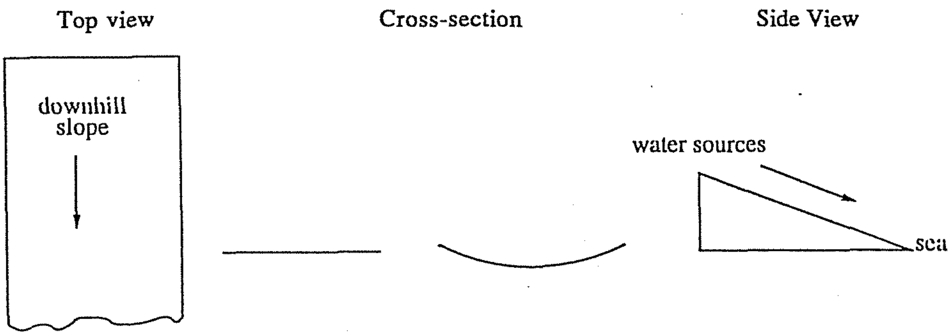
What is the best strategy for packing the two bins?

A selection of strategies was devised and voted on by the students. One of the strategies, but not the best, was "Place all the items in bin A until one won't fit, then revert to bin B until one won't fit, then stop". Although students were keen to suggest strategies for packing the bins they weren't eager to prove their suggestions. The only method offered by them (and their teacher) was that of simulation. "Best strategy" was defined in terms of minimum wastage and after the class had split into groups to perform the various simulations the minimum average wastage obtained was 3.9 u. Ah, but what was the best strategy?

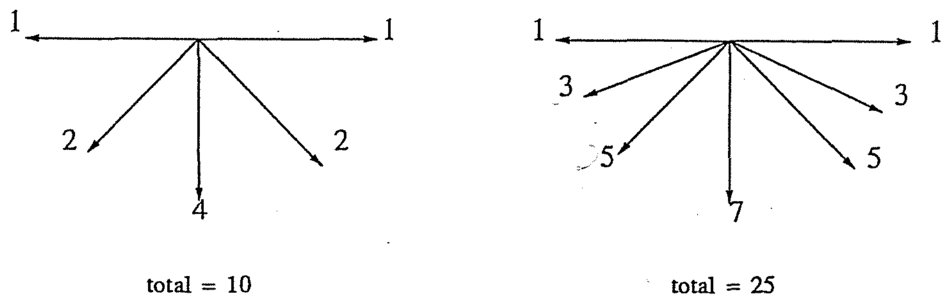
6. Drainage basin

An exercise included in senior geography courses in New Zealand secondary schools is the drainage basin simulation. Here water sources are imagined in a drainage basin and their movement towards the sea, forming streams and rivers as they go, observed and plotted. A sheet of paper takes the place of the drainage basin and it is assumed that the area slopes towards the sea. Other factors are ignored.

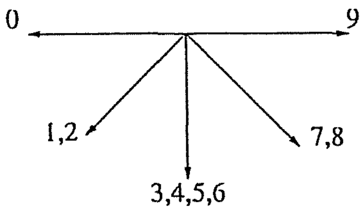
Beginning with 12-20 water sources at the top of the page take each in turn and decide which direction the water travels next. When all the water sources have been dealt with and the movements of the water plotted, the whole process is repeated again many times. Some of the streams move off the page (are lost to the drainage basin), others join up to form rivers. Given enough space, eventually only a few "major" rivers remain and a typical tributary pattern emerges.



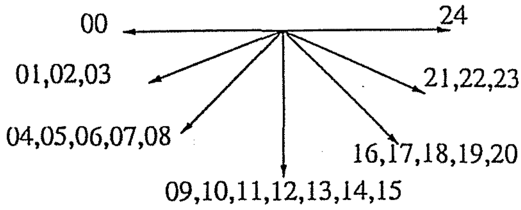
To simulate the movement of the water, a limited number of directions are defined and weighted. Two examples are:



The weightings indicate the likelihood that the water travels in that particular direction, thus, in the right-hand example, the water is seven times more likely to travel down the basin than across it. Numbers are allotted to the directions in proportion to the weightings. These could be:



1-digit numbers



2-digit numbers

Numbers are chosen at random and each stream moved in the direction determined by the weightings. How far can be decided by the students.

There is lots more scope here for students to be involved in the design of the simulation. For example, both the weightings above assume a flat cross-section to the basin but a more realistic valley shape may be chosen (the curved cross-section shown).

The number and range of simulations is only limited by one's imagination and those of the students.

7. Exercise

Design and carry out simulations to solve the following problems.

(i) A platoon sergeant has six men in his platoon. He chooses two at random to accompany him on a mission. After four such missions, how many of his men have been left out altogether?

(ii) Find the probability that two points chosen at random in a unit square are less than 1/2 a unit apart.

(iii) There is a 50-50 chance that a customer arrives at a particular bank teller in a unit time period. If the teller is free the customer is served and this takes two units of time. If not the customer joins or starts a queue. If the teller is free at the outset what is the average queue length after ten time periods?

(iv) From a map of your local region or elsewhere to a known scale, find the area of a lake or other geographical feature.

References

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