Trigonal Spaces with Applications

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1. Introduction

Most of the topics of mathematics that are taught at the public school level begin on an informal basis, quite often in the discrete mode, and sometimes in the context of problem-solving with applications. Since the importance of teaching probability and statistics at public school level seems to be on the way to being well established, one might expect such a humble approach to complement the teaching and learning of these subjects. This does not seem to be the case. Fortunately, one usually acquires some limited knowledge of probability and statistics on an informal basis, but quite often outside the scope and sequence of public school, and on a survival level. Unfortunately, students are usually exposed to severely limited experience, often with discrete aspects, within the school programme, but are then expected to make a quantum jump into probability and statistics based upon continuous and well-formulated mathematics. What follows are some suggestions upon module developments designed to provide problem-solving experiences in developing a sequence of trigonal, numerical arrays, JE(n), which in turn give rise to sequences of discrete, fair probability spaces, P(m(JE(n))), with practical suggestions for open-ended model building. Hopefully these module developments will give some insight into teaching and learning some aspects of problem-solving, as well as naturally generating probability spaces which are discrete and fair. The intent is that the module will contribute to the student's overall ability to understand probability and statistics at a more mature level of sophistication. These ideas were initiated about 1983 by the author, have been field tested with two groups of fifth grade students successfully, and many teachers at the public school level and university level have benefited from workshops based upon these modules.

2. Formulating the sequence of modules: JE(k) $\rightarrow$ P(m(JE(k))

(i) Each module begins with a Spider/Fly, [#T(S$\rightarrow$F) - JE(n)]. Each Spider/Fly should be preceded by a statement of the problem: "There is a spider, S, and a fly, F, such that S travels *on the web from node to node, always downward* (not retracing steps). Suppose n branches lead from each node. How many different trips, #T, can the spider take in getting from the starting node to the fly, (S→F)?" In each problem assume the spider is at the starting node of the web. By letting the fly be at each node, and recording the number of trips S can take to each node, a trigonal array of numbers, JE(n), is generated. Dividing the entries of row m by the sum of that row of entries generates a trigonal array of fair, discrete probability spaces, P(m(JE(n))).

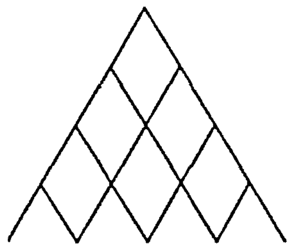
(ii) A description of an appropriate model and events follows the trigonal array of discrete, fair probability spaces. Most of these models are put into a brief historical setting with no intention of completeness. Each of the chosen models can be adapted to fit any of the other modules and are not intended to be comprehensive. There is a characteristic worth mentioning, however. Each of the applications of the models are in terms of replicas of multiples. But as one gains experience with the probability spaces and models one can rather easily consider appropriately identified subdivisions. The primary reason these are not explicitly expressed is that they appear to be somewhat contrived.

(iii) After the theoretical trigonal array of discrete, fair probability spaces is established and an appropriate model is selected, then events are described implementing the selected model and experiments are proposed, which are then graphed and discussed.

Because the primary interest in this paper is the establishing of the sequence of trigonal numeric arrays, JE(n), and the resultant discrete, fair probability spaces, P(m(JE(n))), and because of space limitations, there is virtually no discussion upon the usual experiments associated with these models, or on the distribution functions etc. which usually follow and are useful in developing statistics informally. Only P(m(JE(2))) and P(m(JE6))) will be discussed in any detail.

Example 1: JE(2) : two branches from each node

(#T(S→F) - JE(2))
JE(2)
S



F(#T=4)

R	Sum		P(JE(2))	Sum
0	2^0	P(0(JE(2)))	$\frac{1}{2^0}$	1
1	2^1	P(1(JE(2)))	$\frac{1}{2^1}$ $\frac{1}{2^1}$	1
2	2^2	P(2(JE(2)))	$\frac{1}{2^2}$ $\frac{2}{2^2}$ $\frac{1}{2^2}$	1
3	2^3	P(4(JE(2)))	$\frac{1}{2^3}$ $\frac{3}{2^3}$ $\frac{3}{2^3}$ $\frac{1}{2^3}$	1
4	2^4	P(5(JE(2)))	$\frac{1}{2^4}$ $\frac{4}{2^4}$ $\frac{6}{2^4}$ $\frac{4}{2^4}$ $\frac{1}{2^4}$	1

Events: Consider a bank of m circular spinners with each spinner divided into two semicircles marked with a 1 or 2. m(JE(2)) will describe an event of spinning m such spinners and summing the numbers where each of the spinners are pointing. P(m(JE(2))) will describe all such probabilities of such events. Traditionally, m(JE(2)) is determined by coins and, historically, Pascal made use of a well-known numerical array, the Ancient Chinese Trigonal Array, to introduce the notion of mathematical induction and resolve some gambling problems. So much for the more formal roots of probability and statistics. In some circumstances coins and gambling are not appropriate in public classrooms, and there are also various clues that P(m(JE(2))) had more primitive roots in the use of bones and sticks etc. in making decisions or predictions.

Events and experiments based upon P(m(JE(2))) and graphs:

P(0(JE(2))): The only possible sum with 0 spinners is 0, so the sum of 0 occurs every time the event is tried, $P(S = 0) = 1/2^0 \approx 1$.

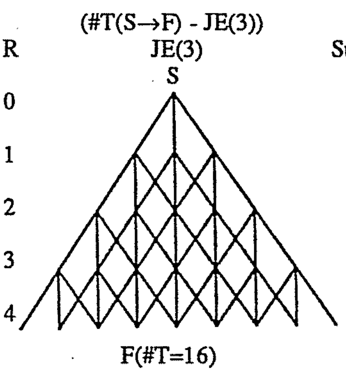
P(1(JE2))): The possible sums are 1 or 2 and each occurs in just one way, so $P(1(JE(2))) : P(S = 1) = 1/2^1, P(S = 2) = 1/2^1$.

P(2(JE(2))): The possible sums, S, are 2, 3, or 4, and each occurs in the following ways:

	S =	2	3	4	Total
			2+1		
		1+1	1+2	2+2	
	#S	1	2	1	2^2
P(2(JE(2))):	P(S)	$\frac{1}{2^2}$	$\frac{2}{2^2}$	$\frac{1}{2^2}$	

Proceeding in this way one can build up the sequence of spaces P(m(JE(2))) corresponding to a binomial experiment with m trials and P(success) = 1/2. As mentioned, students can draw graphs of the probabilities and engage in other discussions of the resulting distributions.

Example 2: JE(3) : three branches from each node

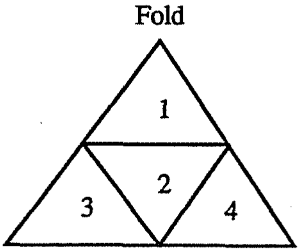


R	Sum		P(JE(3))	Sum
0	3^0	P(0(JE(3)))	$\frac{1}{3^0}$	1
1	3^1	P(1(JE(3)))	$\frac{1}{3^1}\ \frac{1}{3^1}\ \frac{1}{3^1}$	1
2	3^2	P(2(JE(3)))	$\frac{1}{3^2}\ \frac{2}{3^2}\ \frac{3}{3^2}\ \frac{2}{3^2}\ \frac{1}{3^2}$	1
3	3^3	P(3(JE(3)))	$\frac{1}{3^3}\ \frac{3}{3^3}\ \frac{6}{3^3}\ \frac{7}{3^3}\ \frac{6}{3^3}\ \frac{3}{3^3}\ \frac{1}{3^3}$	1
4	3^4	P(4(JE(3)))	$\frac{1}{3^4}\ \frac{4}{3^4}\ \frac{10}{3^4}\ \frac{16}{3^4}\ \frac{19}{3^4}\ \frac{16}{3^4}\ \frac{10}{3^4}\ \frac{4}{3^4}\ \frac{1}{3^4}$	1

Events: Consider m decks of cards, each with 3 cards, and each card numbered 1, 2, or 3. m(JE(3)) will describe an event of shuffling each of the m decks separately and then drawing one card from each of the decks and summing the drawn cards. P(m(JE(3))) will describe all such probabilities of such events. It is assumed that cards will be drawn from decks face down. It is interesting that certain sects of people, such as the Moravians, used similar devices such as three differently marked bones in a clay bowl, as late as the 18th century. One bone was yea, one nay, and then no decision.

Example 3: JE(4) : four branches from each node

Events: We really don't need to create an experimental model as that has been done historically. One of the earliest, in written history, tools used by primitive man for forecasting events, making decisions, and playing games, were Astragalus Bones. Astragalus Bones were heel bones of running animals, recorded about 4500BC, having 4 numbered faces with a couple of unmarked rounded faces. A fair model of such might be a Regular Tetrahedron marked 1, 2, 3, or 4. So an event will be casting m Astragalus Bones and summing the faces that each bone rests upon.

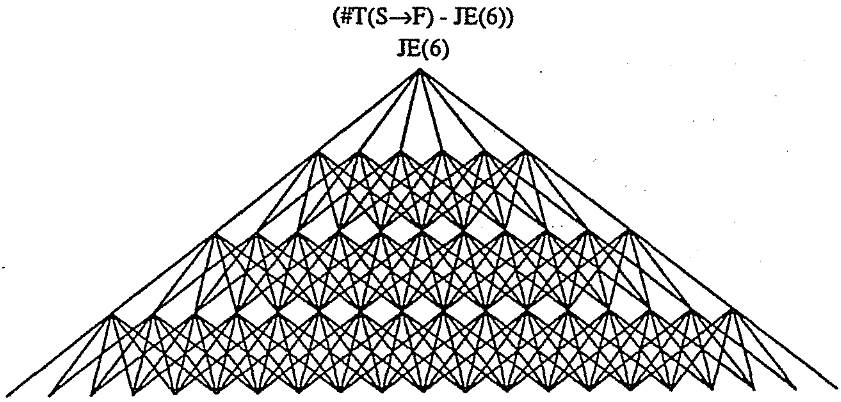


A regular tetrahedron

Example 4: JE(5) : five branches from each node

Events: Consider an urn, an opaque plastic drinking cup which will contain 5 beads, each of which is different in colour only, as a model. m(JE(5)) will describe an event of shaking each of the m urns separately and then drawing, without looking, a bead from each cup. Each of the coloured beads are assigned numbers by colour of 1, 2, 3, 4, or 5. The sum of the m drawn beads completed the event. Many similar models are in practice today in games.

Example 5: JE(6) : six branches from each node



R	JE(6) S	Sum
0	1	6^0
1	1 1 1 1 1 1	6^1
2	1 2 3 4 5 6 5 4 3 2 1	6^2
3	1 3 6 10 15 21 25 27 27 25 21 15 10 6 3 1	6^3

P(JE(6))

P(0(JE(6)))	$\frac{1}{6^0}$	1
P(1(JE(6)))	$\frac{1}{6^1}$ $\frac{1}{6^1}$ $\frac{1}{6^1}$ $\frac{1}{6^1}$ $\frac{1}{6^1}$ $\frac{1}{6^1}$	1
P(2(JE(6)))	$\frac{1}{6^2}$ $\frac{2}{6^2}$ $\frac{3}{6^2}$ $\frac{4}{6^2}$ $\frac{5}{6^2}$ $\frac{6}{6^2}$ $\frac{5}{6^2}$ $\frac{4}{6^2}$ $\frac{3}{6^2}$ $\frac{2}{6^2}$ $\frac{1}{6^2}$	1
P(3(JE(6)))	$\frac{1}{6^3}$ $\frac{3}{6^3}$ $\frac{6}{6^3}$ $\frac{10}{6^3}$ $\frac{15}{6^3}$ $\frac{21}{6^3}$ $\frac{25}{6^3}$ $\frac{27}{6^3}$ $\frac{27}{6^3}$ $\frac{25}{6^3}$ $\frac{21}{6^3}$ $\frac{15}{6^3}$ $\frac{10}{6^3}$ $\frac{6}{6^3}$ $\frac{3}{6^3}$ $\frac{1}{6^3}$	1

Events: Historians tend to believe that dice evolved from Astragalus Bones. These six-faced heel bones of running animals had, in their natural state, four flat faces which were numbered and probably were examples of the earliest form of discrete probability models. Over the years people tended to grind off the two rounded faces so that the bones evolved in six flat numbered faces, the earliest models of dice. Much later, the regular hexahedral numbered dice of today evolved. Dice of today are usually designed to be fair and the opposite faces are marked such that the sum is seven. An event will be rolling m dice and summing the faces that each die shows. A fair die, regular hexahedron, where each face is a regular quadragon, can be constructed from the pattern illustrated on the right.