Students' intuition and doing mathematics. An example in probability

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Abstract.

If it is accepted that the concepts of probability are complicated, it should be also accepted that they are very near to the daily life of common people. Anyway, everybody has to face a variety of situations of uncertainty that can cause either anxiety or joy. As the idea is to teach these concepts to the students, the best way to do is it to have fun when carrying it out. This paper reports the experience with a group of students who are preparing to become high school teachers, in the world of probability by talking about soccer. With this sport as a reference a question is posed such that when students are asked about it, it not only allows an interesting probabilistic analysis, but also takes them, when solving it, to other mathematics concepts like limits and derivatives. The whole situation is presented: its position as conjecture, attempts of answering that include computer work and graphics up to its formal proof.

1. Introduction

Some of the answers given for the researchers to the question: how to present some mathematics topics and how to develop them in the classroom? appear, by instance, in the Principles and Standards for School Mathematics: Discussion Draft (NCTM, 1998). Some of the proposals included in it are:

Mathematics are learned through (and for) solving problems.
Mathematics involves trial and reasoning.
Problems related to students' interests stimulate and facilitate the acquisition of the knowledge.
The idea of proposing problems that allow the connection of several areas of the mathematics must be promoted.
Group activity and interaction stimulates learning.
Students' conjectures and attempts to solve the problems must receive attention from teachers.

In this work, the experience of the author as a class teacher of a group of twenty students (between 18 and 20 years old), in their first course of statistics of the third semester of the major in Mathematics for high school teachers is presented. Having the ideas mentioned in the Principles and Standards for School Mathematics in mind and with the use of a software, developed by the author, which presents games and real situations familiar to the students, some problematic situations are generated. These situations allow students to think about the basic elements that conform the theory of probabilities before the theoretical aspects are presented. Thus, in the very first class of the course, based on an activity around the classical idea of probability and before the necessity of finding a solution to the questions presented by the software, a solution is accepted, at first sight by all the members of the group, including the teacher, but one student. Since the idea is not to ask for justifications for a negative answer but to justify an affirmation, the search for the answer led us to use algebra and calculus to solve the problem.

The story presented below is the summary of several classroom sessions (working the students and the teacher together) and homework done by the students alone.
The work is organised as follows: in number 2 the problematic situation is presented, in number 3 the answers given by the students and the follow up activity is described and, finally, in number 4 some conclusions are discussed.

2. Context

In an on-line educational module (OEM) with interactive characteristics designed to familiarise Mathematics students, who are preparing to become high school teachers, with the world of probability, the following case is presented: Imagine that Colombia and Brazil play a soccer match in the Soccer World Cup (the sport tolerates the illusions), and that it is time to decide which team plays ball at first. With this purpose, the referee flips a coin, previous agreement that if it is heads Colombia starts, and that if it is tails, Brazil is the starter. The student is asked if he considers the procedure to be fair and to state a reason for his answer. With the purpose of securing this first idea of probability in equally likely events as the ratio between the number of possibilities in favour and the total number of possibilities, the beginning setup is changed supposing that the referee uses a die instead of a coin. Taking into account just the number of the upper face of the die the student is asked to design a way that independently satisfies each of the following conditions:

That the election is fair.
That the probability for Colombia to start is two times that of Brazil.
That the probability for Colombia to start is three times that of Brazil.

3. Answers and their consequences

For the first two conditions the students gave a variety of valid and expected answers. For the third and most interesting one, students gave answers that motivated larger discussions, and even the writing of this article. The answers to the latter condition can be grouped in three kinds:

1. An answer that coincides with the objective that the authors of the module intended which responds to an argument as the following one*: If the die is thrown one time, let x be the number of results in favour of the Brazil and 3x the number of results in favour of the Colombia. As the total of results of the die is 6, you get to the equation 4x = 6, which does not have an integer solution indicating that it is not possible to design a situation that holds this requirement.* This solution or better said the 'no solution' was outlined by very few students who dared to believe that a problem proposed by the teacher could not have a solution. The remaining of the students that perceived that the problem did not have solution with one throwing of the die, proceeded to throw it twice or three times for giving solutions like the 2 below.

2. Answers associated to several rolls of the die characterised by misinterpretation of the equally likely events, like the following:

(a) *A die is thrown 2 times and the results are added. As different possible results are 11, it is impossible to carry out the division requested.* The argument of impossibility that they used in this moment and that they did not explicit for one throwing of the die, did not show up algebrally as in 1 above. Perhaps the fact that the number 11 is a prime number it was an enough reason against any division that they intended.

(b) *A die is rolled three times and results are added. As different possible results are 16, it is enough to assign 4 any different results to Brazil and the remaining 12 to Colombia*.

3. The attempts for solution with 'common sense' that generate the approval of the rest of students with words like 'it is logical' or 'of course', and which are sometimes not clear neither to support them nor to contradict them, but which evidently are the ones that generate discussion, because they break up the

natural state of things in the classroom, forcing the teacher to share his reasoning with the students in the search of an answer that contrarily to most cases, he really does not know. The following solution is of this type of answer : *The captain of the Brazilian team rolls the die once and the referee writes down the number got. After that, the captain of the Colombian team rolls the die 3 times and the referee writes down the highest number got. The match is started by the one who has got the highest value. The solution is justified by the fact that the one who rolls 3 times has 3 times probability to win. 'It's logical'.* When this situation occurs, the teacher and the students, except Ludwig, accepted the 'logical' of the answer. Ludwig said:

Ludwig: *I am sorry, but I do not agree with you*.
Teacher: *Why?*
Ludwig: *I do not know, but I am not as sure as you are.*
Teacher: *OK Guys, we have to prove it in order to convince Ludwig.*

The same module taken as reference includes after this and some other similar activities, a round activity where the different given answers are confronted and the different interpretations are discussed. Respect to the answers of the numeral 2 (a) above, just by establishing that instead of throwing 2 times the die, you throw two different coloured dice simultaneously, the students got the idea that the equally likely results are not the sum of the numbers obtained in the dice, but the pairs formed considering the result of the first die as the first component and the result of the second die as the second component. The starting question is settled once again here after this observation, and various right answers are obtained throwing the die twice. Keeping this idea in mind, the answer given when throwing the die 3 times is quickly denied. An assignment of the sum of the values of the 3 dice to any of the two teams that allows us to obtain the wanted ratio of 3 to 1 does not represent a conceptual challenge to the students, even though it demands the elaboration of an enormous chart enumerating all possible results (216) and their respective sums.

On the contrary, the third solution is no longer easy to be refused or proved. With the experience gained solving the previous cases, the first attempt for solution is to numerate all the possible results that are obtained when Colombia throws the die three times, to calculate the highest result and to compare it with the result obtained by Brazil. Later on, to count the results in favour the Colombia and the ones in favour of Brazil, and calculate the ratio. As the number of possible results is as the number of possible results is $6^4 = 1296$, (not a very stimulating figure to enumerate all cases), the students were given the task to count through different ways the possibilities for each team to win and the number of ties, looking for the way that allows them to extrapolate to any quantity of throws. After many attempts and after a lot of discussion the teacher and the students got to the following counting strategy:

1. The number of ties corresponds to the number of possible results of the Colombian team. As Colombia throws 3 times, the possible results are 6^3. This statement is justified by the fact that Brazil can tie if it obtains in a single throw the highest number got by Colombia.
2. The number of victories of Brazil is obtained by considering one by one its possible results and counting the results in which Colombia loses. This way, the following table is obtained:

Results of throws of Brazil	Number of times that Colombia loses
1	0
2	1 x 1 x 1
3	2 x 2 x 2
4	3 x 3 x 3
5	4 x 4 x 4
6	5 x 5 x 5
Total victories Of Brazil	$1^3 + 2^3 + \ldots + 5^3$

In order to find the number of possibilities in favour of Colombia, it is enough to carry out a subtraction:

RFC = Total - RFB – T where

RFC : Results in favour of Colombia;
RFB : Results in favour of Brazil
T : Ties

Thus, we Obtain RFB = 225, RFC = 855 and the wanted ratio is $\frac{RFC}{RFB} = 3.8$

"That means that it is not 3 as it was expected". "What happens with the ratio of possibilities if the ratio between throws is different to 3?", were some of the comments that arose after the result.

Taking profit out of the fact that the counting strategy allowed immediate generalizations, the following table was made with the purpose of finding an answer to a most general problem, i.e., calculating the ratio of possibilities when the number of throws of Colombian team is higher than 3.

Ratio between throws	Ratio between Probabilities
2	2.27
3	3.80
4	5.62
5	7.79
6	10.04
7	13.5
8	17.1
9	21.5
10	26.8

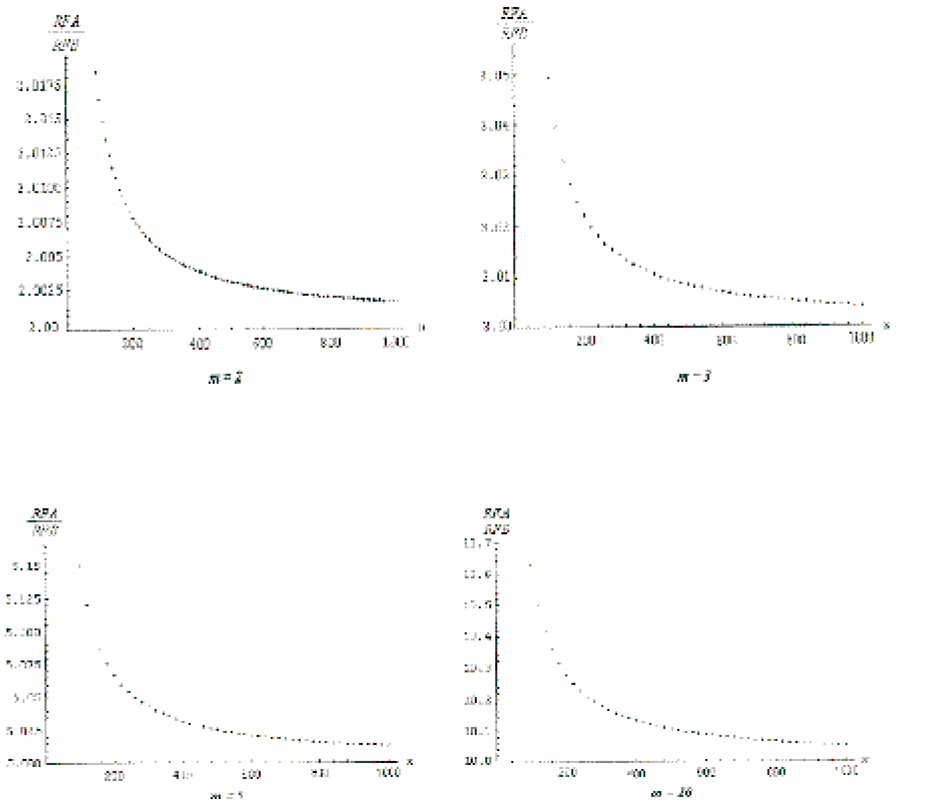
Some of the answers to the question: *What does the table tell us?,* were the following:

1. That in all cases, the ratio of possibilities is always higher than the ratio between throws.
2. That if we divide the ratio between probabilities – column B – by the ratio between throws – column A – the quotient increases as we increase the number of throws.
3. In fact, "*what we had believed is not true, and gets even worse as we increase the number of throws"*

Keeping this experience in mind, in the activity of discussion in the class another situation that had not been considered was stated: increasing the number of faces of the die. Will it be possible that with a die that has more or perhaps less than 6 faces, the ratio of throws is similar to the ratio of possibilities? "*It could be true, but let us do it with a computer to avoid making so many operations",* the students suggested.
If we suppose that there exist dice with any number of faces in such a way that all the possible results are equally likely, we want to find a complete answer to the following question:

Suppose that exist a dice with any number, n, of faces equally likely. Likewise suppose that two players, A and B, accept to play a game where the one who gets the higher value is the actual winner, knowing that player A throws the die m times and player B just a single time. Are there values for n and m such that the probability for player A to win be m times the probability for player B to win?

Consequently we developed a program in Mathematica that allowed calculating the ratio between the results in favour to each player for different values of *n* (number of faces of the die) and *m* (number of throws of player A). The results for particular cases *m* = 2,3,5,10 and different values of *n,* are shown in the following graphics.



What do the graphics tell us? Students just could not wait to answer.

1. *"That for a given ratio between throws, the ratio between the respective probabilities of the two players decreases with the increase of the number of faces of the die".*
2. *"The ratio between the probabilities is much more similar to the ratio between throws as one increases the number of faces of the die".*

Up to this point it could have been considered quite enough if it had not been taken into account that at the time of the experience students had already carried out basic course in Mathematics: High Algebra and Differential Calculus. For this

reason it was considered pertinent to try a formal proof of the fact observed in the previous figures.
Actually, the idea was to prove that

(1) $$\lim_{n\to\infty} \frac{RFA}{RFB} = m$$

where RFA are the results in favour of player A, RFB the results in favour of player B, *n* the number of faces and *m* the number of throws of the player A.
Based on the results obtained in the case of three throws of player A, it was known that

(2) $$RFB = 1^m + 2^m + \ldots + (n-1)^m$$

And that

(3) $$RFA = n^{m+1} - n^m - RFB$$

Which implies that the problem of limit (1) was reduced to calculate the addition

(4) $$1^m + 2^m + \ldots + (n-1)^m$$

for any *m* and *n* natural numbers.

For particular cases m = 1,2 in (4) students remembered that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

from which they could check expression (1) for these cases and deepen their confidence on the generality of the statement.

Although at the beginning the goal was to find a general expression for the sum of expression (4), this trial has be aborted because of its difficulty. Consequently, it was thought to look for an expression that worked for calculating the limit expressed in (1). Thus, the following expression was found:

(5) $$1^m + 2^m + \ldots + n^m = \frac{n^{m+1}}{m+1} + P(n)$$

where P(*n*) is a polynomial in *n* whose degree is less or equal that *m.* The students proved (5) by induction for any exponent *m* and any natural number *n.*
Replacing (2), (3) and (5) in (1) we obtained

$$\lim_{n\to\infty} \frac{RFA}{RFB} = \lim_{n\to\infty} \frac{n^{m+1} - n^m - \dfrac{n^{m+1}}{m+1} - P(n)}{\dfrac{n^{m+1}}{m+1} + P(n)} = m$$

after dividing both the numerator and denominator by n^{m+1} to suppress the indetermination. So we get to a happy ending or the arisen concern. However, as all good Math classes, things just could not end this way. It was left as homework the task (i) to demonstrate that expression RFA/RFB considered as function of *n*, is decreasing as long as *n* increases as it had been perceived in the graphics by the students: (ii) Find the limit of the expression (1) when *m* (the number of throws) is who goes to infinity.

4. Conclusions

Out of this experience one gets several ideas that can be useful in teaching processes to students who are just beginning to study probability:

1. Stating familiar situations to students, like soccer in this article, makes them enthusiastic to participate.
2. If we stimulate and permit all kinds of answers without interfering in their elaboration, it is very possible that the student's logic and intuition arise and that interesting problems show up.
3. The complete approach to problems posed by the students questions permits as in this example, to set out a result in a general way and to prove it by using concepts already worked by them in previous courses.
4. The use of the technology that allows to do the tedious calculus and doing graphics to see the results.

Finally, it is noteworthy to stress the fact of that the teacher did not know the answer to the students' question when they were posed. This situation allowed a real mathematics work by the teacher who had to attempt and make mistakes many times in front of his students while he used different strategies of solving problems.

REFERENCES

NCTM (1998). Principles and Standards for School Mathematics: Discussion Draft. Virginia, National Council of Teachers of Mathematics.