Students’ Biases in Conditional Probability Reasoning

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INTRODUCTION

- Conditional probability and Bayesian reasoning are important components in undergraduate statistics and statistical literacy
- However, several studies show psychological biases related to this concept
- Most of these biases have been studied in isolation, so it is unclear whether these incorrect types of reasoning are related to each other or to formal mathematical knowledge
- The purpose of our investigation is exploring possible relationships between the biases described in the literature and whether the biases decrease with instruction

CONDITIONAL PROBABILITY REASONING

- Pioneer study of Fischbein and Gazit (1984) with children (10-12 year-olds)
- Tarr and Jones (1997) identified four levels of thinking about conditional probability (9-13 year-old students):
  1. Subjective: Students use subjective reasoning.
  2. Transitional: Students demonstrate some recognition of whether consecutive events are related or not.
  3. Informal quantitative: imprecise differentiation of “with and without replacement situations” and quantification of probabilities;
  4. Numerical: Students state conditions for two events to be related, assign the correct numerical probabilities and distinguish between dependent and independent events.

BIASES RELATED TO CONDITIONAL PROBABILITY REASONING

- Even when students progress towards the upper level, difficulties remain at high school and university (Tarr & Lannin, 2005)
- *Confusing causal and diagnostic situations:*
  - The person who assesses *P(A/B)* perceive different relationship between *A* and *B* depending on the context (Tversky and Kahneman, 1982a).
  - If *B* is perceived as a cause of *A, P(A/B)* is viewed as a causal relation, and if *A* is perceived as a possible cause of *B, P(A/B)* is viewed as a diagnostic relation.
- *Fallacy of the transposed conditional:* people confuse P(A/B) and P (B/A) (Falk, 1986).

BIASES RELATED TO CONDITIONAL PROBABILITY REASONING

- *The fallacy of the time axis:*
  - Students are reluctant to believe that an event could condition another event that occurs before it (Falk, 1989)

- *The base rate fallacy:* Tversky and Kahneman (1982a)
  - In Bayes' problems you are given statistics for a population as well as for a particular part of the population
  - Both types of information has to be considered together to solve the problem; however, people tend to ignore the population base rate (Bar-Hillel, 1983; Koehler, 1996).

BIASES RELATED TO CONDITIONAL PROBABILITY REASONING

- *Conjunction fallacy*
  - People's unawareness that a compound probability cannot be higher than the probability of each single event (Tversky and Kahneman, 1982a)
- Confusion between *independence* with *mutually exclusiveness* (Sánchez, 1996; Truran & Truran, 1997).
- *Synchronic and diachronic situations:* individuals may perceive them differently. (Sánchez and Hernández, 2003).
  - Diachronic situations: series of sequential experiments.
  - Synchronic situations: static, do not incorporate an underlying sequence of experiments.

Other research

- *Other research*
- *Frequency versus probabilistic format:*
  - (Gigerenzer, 1994; Gigerenzer & Hoffrage, 1995; Sedlmeier, 1999; Martignon, & Wassner, 2002).
- Teaching experiments with resource to software (Díaz & Batanero, 2007; *n*=75 psychology students).
- Variables and strategies in simple conditional probability problems (several papers by Lonjedo & Huerta)

METHOD

- We built the CPR questionnaire to simultaneously assess formal understanding and biases related to conditional probability (Díaz & de la Fuente, 2006, 2007).
- The questionnaire was given to different samples of students majoring in Psychology (18-19 year-olds).
- Analysis
  - Factor analysis of the set of responses in the CPR questionnaire by 404 students (Granada, n=208 students and Murcia n=196 students).
  - Discriminant analysis to compare the performance of the 208 students from Granada (with instruction) with those of another group (177 students from the universities of Huelva and Jaen) who took the questionnaire before studying conditional probability.

RESULTS: STRUCTURE OF RESPONSES

- We identified 7 factors with eigenvalues higher than 1 that explain 59% of the total variance:
  - 21% the first factor,
  - 7 % the second factor,
  - About 6% in each of the remaining factors.
- This confirms the specificity of each item, and the multidimensional character of understanding conditional probability.
- It also reveals the higher relevance of the first factor

Item	Factor						
	1	2	3	4	5	6	7
18. Solving Bayes' problem	.77						
14. Solving total probability problem	.76						
17. Solving product rule problem, in case of dependence	.76						
16. Solving product rule problem, in case of independence	.67						
15. Solving a conditional probability problem, in case of independence	.43		.42				
12. Describing the restricted sample space	.40		.46				
2. Base rate fallacy	.34						.48
1b. Computing conditional probability from a 2- way table	.32	.61					
1c. Computing joint probability, from a 2- way table		.79					
1a. Simple probability, from a 2- way table		.61					
1d. Computing inverse conditional probability from a 2- way table		.77					
13. Solving a conditional probability problem, in a single experiment			.67				
11. Defining conditional probability and giving an example			.59				
4. Solving a conditional probability problem, in case of dependence			.39		.44		
9b. Time axis fallacy				.71			
8. Time axis fallacy				.70			
10. Solving a joint probability problem in diachronic experiments				.35			-.46
5. Computing conditional probability from joint and compound probability					.66		
6. Conjunction fallacy					.62		
7. Transposed conditional /causal-diagnostic						-.65	
9a. Computing conditional probability, dependence						.66	
3. Independence /mutually exclusiveness							.68

RESULTS:
STRUCTURE OF RESPONSES

1. *Ability to solve complex conditional probability problems:* most of the open-ended problems, (Bayes, total probability and compound probability problems). Solving any of these problems requires two stages, in the first of which a conditional probability is computed.
2. Computing simple, joint and conditional probability from a two-way table (item 1), probably the *task format* affected performance, also noticed by Ojeda (1996), Gigerenzer (1994) and Lonjedo and Huerta (2005).
3. *Level 4 reasoning* in the classification by Tarr and Jones (1997)

- The remaining factors point out to the different biases affecting conditional probability reasoning.
- The different biases (transposed conditional, time axis fallacy, conjunction fallacy, independence/mutually exclusiveness) appear in different factors

RESULTS: IMPROVEMENT WITH INSTRUCTION

- Discriminant analysis is used to study improvement after instruction
  - Wilks' Lambda= 0.63; Chi-Square $\chi 2$=171.117;
  - Canonical correlation=0.697;
  - 82.3% students correctly classified.
- Differences always favour the group with instruction and are statistically significant, with exception of the items assessing
  - Conjunction fallacy,Fallacy of transposed conditional, Difference between causal and diagnostic reasoning, Fallacy of time axis.
- The CPR questionnaire may serve to discriminate between students with and without specific instruction if we exclude part of the items assessing psychological biases.

Item		No instruction (n=177)	Instruction (n=208)	p-value
1b.	Computing conditional probability from a 2- way table	67	94	0.000
1c.	Computing joint probability, from a 2- way table	29	63	0.000
1a.	Simple probability, from a 2- way table	35	69	0.000
1d.	Computing inverse conditional probability from a 2- way table	37	70	0.000
2.	Base rate fallacy	33	53	0.000
3.	Independence /mutually exclusiveness	23	41	0.000
4.	Solving a conditional probability problem, in case of dependence	77	89	0.001
5.	Computing conditional probability from joint and compound probability	37	48	0.042
6.	Conjunction fallacy	21	24	0.465
7.	Transposed conditional /causal-diagnostic	35	35	0.989
8.	Time axis fallacy	8	13	0.142
9a.	Computing conditional probability, dependence	72	81	0.050
9b.	Time axis fallacy	37	25	0.009
10.	Solving a joint probability problem in diachronic experiments	62	76	0.002
11.	Defining conditional probability and giving an example	17	47	0.000
12.	Describing the restricted sample space	56	67	0.050
13.	Solving a conditional probability problem, in a single experiment	20	33	0.005
14.	Solving total probability problem	18	55	0.000
15.	Solving a conditional probability problem, in case of independence	34	75	0.000
16.	Solving product rule problem, in case of independence	25	56	0.000
17.	Solving product rule problem, in case of dependence	24	59	0.000
18.	Solving Bayes’ problem	20	49	0.000

RESULTS: IMPROVEMENT WITH INSTRUCTION

- Important improvement in performance in all open-ended problem solving tasks:
  - 13 (conditional probability, in a single experiment),
  - 14 (total probability),
  - 15 (conditional probability, independence),
  - 16 (product rule, independence),
  - 17 (product rule, dependence),
  - 18 (Bayes' problem).
- Other items with good discrimination
  - 1 (computation of probabilities from a 2-way table),
  - 11 (defining conditional probability)
  - 2 (base rate fallacy)
  - 3 (distinguishing between independence and mutually exclusiveness).

CONCLUSIONS

- Students' performance in problem solving ability and formal understanding of conditional probability improved with instruction;
- However, some of the biases described were widespread and did not improve with instruction.
- Factor analysis reflect the complex relationship between probabilistic concepts and intuition (Borovcnik, Bentz & Kapadia, 1991; Borovcnik & Peard, 1996): items assessing biases were unrelated to formal knowledge.
- These results are consistent with Batanero, Henry and Parzysz (2005) who trace an analogous lack of intuition in the historical development of the discipline.

CONCLUSIONS

- Need for reinforcing the study of conditional probability in teaching data analysis at university level and changing teaching approaches.
- Students should be "given greater motivation to attend closely to the nature of the inferential tasks that they perform and the quality of their performance" (p. 280) and consequently "statistics should be taught in conjunction with material on intuitive strategies and inferential errors" (p. 281). Nisbett and Ross (1980)
- Items included in the CPR questionnaire may serve to assess the extent of these biases among their students or to build up didactic situations

THANKS FOR YOUR ATTENTION

THE QUESTIONNAIRE

Item 1. (Estepa, 1994). In a medical centre a group of people were interviewed with the following results:

	55 years-old or younger	Older than 55	Total
Previous heart stroke	29	75	104
No previous heart stroke	401	275	676
Total	430	350	780

Suppose we select at random a person from this group:

a. What is the probability that the person had a heart stroke?
b. What is the probability that the person had a heart stroke and, at the same time is older than 55?
c. When the person is older than 55, what is the probability that he/she had a heart stroke?
d. When the person had a heart stroke, what is the probability of being older than 55?

THE QUESTIONNAIRE

Item 2. (Tversky & Kahneman, 1982a). A witness sees a crime involving a taxi in a city. The witness says that the taxi is blue. It is known from previous research that witnesses are correct 80% of the time when making such statements. The police also know that 15% of the taxis in the city are blue, the other 85% being green. What is the probability that a blue taxi was involved in the crime?

a. $\frac{80}{100}$

b. $\frac{15}{100}$

c. $\frac{15}{100} \times \frac{80}{100}$

d. $\frac{15 \times 80}{85 \times 20 + 15 \times 80}$

Item 3. (Sánchez, 1996). A standard deck of playing cards has 52 cards. There are four suits (clubs, diamonds, hearts, and spades), each of which has thirteen cards (2,..., 9, 10, Jack, Queen, King, Ace). We pick a card up at random. Let A be the event "getting diamonds" and B the event "getting a Queen". Are events A and B independent?

a. A and B are not independent, since there is the Queen of diamonds.
b. A and B are only then independent when we first get a card to see if it is a diamond, return the card to the pack and then get a second card to see if it is a Queen.
c. A and B are independent, since P(Queen of diamonds)= P(Queen) $\times$ P(diamonds).
d. They A and B are not independent, since P(Queen | diamonds) $\neq$ P(Queen).

THE QUESTIONNAIRE

Item 4. There are four lamps in a box, two of which are defective. We pick up two lamps at random from the box, one after the other, without replacement. Given that the first lamp is defective, which answer is true?

a. The second lamp is more likely to be defective.

b. The second lamp is most likely to be correct.

The probabilities for the second lamp being either correct or defective are the same

Item 5. (Eddy 1982). 10.3 % of women in a given city have a positive mammogram. The probability that a woman in this city has both positive mammogram and a breast cancer is 0.8%. A mammogram given to a woman taken at random in this population was positive. What is the probability that she actually has breast cancer?

a. $\frac{0.8}{10.3} \times = 0.0776,\ 7.76\,\%$ b. $10.3 \times 0.8 = 8.24,\ 8.24\,\%$ c. $0.8\ \%$

Item 6. (Tversky & Kahneman, 1982 b). Suppose a tennis player reaches the Roland Garros final in 2005. He has to win 3 out of 5 sets to win the final. Which of the following events is more likely or are they all equally likely?

a. The player will win the first set.

b. The player will win the first set but lose the match.

Both events a. and b. are equally likely

THE QUESTIONNAIRE

Item 7. (Pollatsek et al. 1987). A cancer test is administered to all the residents in a large city. A positive result is indicative of cancer and a negative result of no cancer. Which of the following results is more likely or are they all equally likely?

a. A person has cancer if they get a positive result.

b. To have a positive test if the person has cancer.

The two events are equally likely.

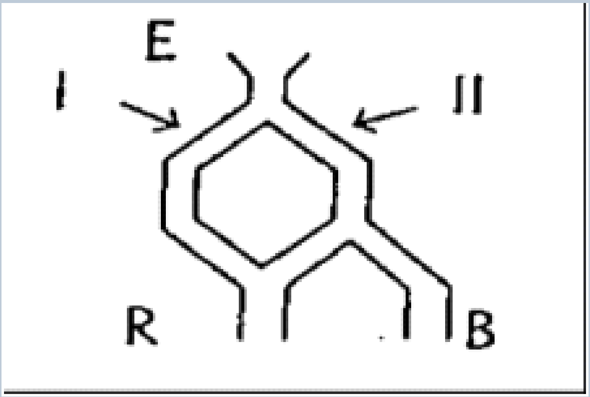
Item 8. (Ojeda 1996).We throw a ball into the entrance E of a machine (see the figure). If the ball leaves the system through exit R, what is the probability that it passed by channel I?

a. $\frac{1}{2}$

b. $\frac{1}{3}$

c. $\frac{2}{3}$

d. Cannot be computed



THE QUESTIONNAIRE

Item 9. (Falk, 1986, 1989). Two black and two white marbles are put in an urn. We pick a white marble from the urn. Then, without putting the white marble into the urn again, we pick a second marble at random from the urn.

If the first marble is white, what is the probability that this second marble is white? $P(W_2 \mid W_1)$

a. $\frac{1}{2}$ b. $\frac{1}{6}$ c. $\frac{1}{3}$ d. $\frac{1}{4}$

If the second marble is white, what is the probability that the first marble is white? $P(W_1 \mid W_2)$

a. $\frac{1}{3}$ b. Cannot be computed c. $\frac{1}{6}$ d. $\frac{1}{2}$

Item 10. An urn contains one blue and two red marbles. We pick up two marbles at random, one after the other without replacement. Which of the events below is more likely or are they equally likely?

a. Getting two red marbles.
b. The first marble is red and the second is blue
c. The two events a) and b) are equally likely.

Item 11. Explain in your own words what a simple and a conditional probability is and provide an example.

Item 12. Complete the sample space in the following random experiments:

a) Observing gender (male/female) of the children in a three children family (e.g. MFM,...)
b) Observing gender of the children in a three children family if two or more children are male.

THE QUESTIONNAIRE

Item 13. In throwing two dice the product of the two numbers is 12.
What is the probability that none of the two numbers is a six (we take the order of the numbers into account).

Item 14. 60% of the population in a city are men and 40% women. 50% of the men and 35% of the women smoke. We select a person from the city at random; what is the probability that this person is a smoker?

Item 15. A person throws a die and writes down the result (odd or even). It is a fair die (that is, all the numbers are equally likely). These are the results after 15 throws:

Odd, even, even, odd, odd, even, odd, odd, odd, odd, even, even, odd, odd, odd.

The person throws once more. What is the probability to get an odd number this time?

Item 16. A group of students in a school take a test in mathematics and one in English. 80% of the students pass the mathematics test and 70% of the students pass the English test. Assuming the two subjects score independently, what is the probability that a student passes both tests (mathematics and English)?

THE QUESTIONNAIRE

Item 17. According to a recent survey, 91% of the population in a city usually lie and 36% of those usually lie about important matters. If we pick a person at random from this city, what is the probability that the person usually lies about important matters?

Item 18. (Totohasina 1982). Two machines M_1 and M_2 produce balls. Machine M_1 produces 40 % and M_2 60% of balls. 5% of the balls produced by M_1 and 1% of those produced by M_2 are defective. We take a ball at random and it is defective. What is the probability that that ball was produced by machine M_1?