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18. Combinatorial Reasoning and its Assessment

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Purpose

In this chapter we provide some answers to the following questions:

- What is combinatorics and what role does it play in teaching and learning probability?
- What components of combinatorial reasoning should we develop and assess in our students?
- Are there any task variables that influence students' reasoning and provoke mistakes when solving combinatorial problems?
- What are the most common difficulties in the problem-solving process? How should we consider these variables in the teaching and assessment of the subject?

We illustrate these points by presenting some examples and test items taken from different research work about combinatorial reasoning and samples of students' responses to these tasks.

WHAT IS COMBINATORICS?

The scope of combinatorics is much wider than simply solving permutation, arrangement, and combination problems. In his "Art Conjectanding," Bernoulli described combinatorics as the art of enumerating all the possible ways in which a given number of objects may be mixed and combined so as to be sure of not missing any possible result. According to Hart (1992), combinatorics is the mathematics of counting. It is concerned with problems that involve a finite number of elements (discrete sets), with which we perform different operations. Some of these operations only modify the set structure (e.g., a permutation of its elements) while others change the set composition (taking a sample). We are usually interested in a *combinatorial configuration* or composition of the result of some of these operations, and we attempt to answer the following questions:

- Does a specific combinatorial configuration exist?

- How many combinatorial configurations are there in a given class?
- Is there an optimum solution to a (discrete) problem?

These questions correspond to three different categories of combinatorial problems: *existence problems* deal with whether a given problem has a solution or not; *counting problems* investigate how many solutions may exist for problems with known solutions; *optimization problems* focus on finding a best solution for a particular problem. Considering Bernoulli's description, we must add *enumeration problems* that correspond to the question of whether we can produce a procedure for systematically listing all the solutions for a given problem.

The teaching of combinatorics is currently not considered necessary by many statistics teachers, probably because they restrict its meaning to counting problems and to combinatorial operation formulae. Nevertheless, in our teaching proposal (Batanero et al., 1994) we have distinguished the following components in the teaching and assessment of combinatorics.

Basic combinatorial concepts and models:

- Combinatorial operations: combinations, arrangements, permutations, concept, notation, formulae;
- Combinatorial models:
  - Sampling model: population, sample, ordered/non-ordered sampling, replacement;
  - Distribution model: correspondence, application;
  - Partition model: sets, subsets, union.

Combinatorial procedures:

- Logical procedures: classification, systematic enumeration, inclusion/exclusion principle, recurrence;
- Graphical procedures: tree diagrams, graphs;
- Numerical procedures: addition, multiplication and division principles, combinatorial and factorial numbers, Pascal's triangle, difference equations;
- Tabular procedures: constructing a table, arrays;
- Algebraic procedures: generating functions.

Most of these contents are also linked to probability. Moreover, we can easily identify relevant statistics and probabilistic questions in each of the aforementioned combinatorial problem categories, as we can observe in the following classical situation of experimental design:

Example 1:

> Suppose you want to assess the effect of two different fertilizers on the improvement of tomato production. You have two types of tomato available and you would like to evaluate simultaneously the effect of low/high humidity degree on the production. Is it possible to design such an experiment using only 4 experimental plots?

In this situation, we can easily identify an example of a combinatorial *existence* problem. When trying to list all the different combinations of factors, we would be dealing with a combinatorial *enumeration* problem. If we ask for the total number of the three-factor combinations, we would be interested in a *counting problem*, for which the solution is 2x2x2=8, because in each factor (tomato plant, fertilizer, humidity) we have only two possible values. Finally, an *optimization* problem can be proposed when asking the number of different two-values factors that could be evaluated with only 4 experimental plots.

We may note in this example some features of many combinatorial problems that Kapur (1970) highlighted:

- Since it does not depend on calculus, it poses suitable problems for different grades.
- Usually very challenging problems can be discussed with students.
- It can be used to train students in enumeration, making conjectures, generalization, optimization, and systematic thinking.
- Many applications in different fields: chemistry, biology, physics, communications, number theory, etc., can be presented. Because of the interconnections between combinatorics and probability, which we shall discuss in the following section, this may serve to show students the applicability of probability in these different subjects.

THE ROLE OF COMBINATORICS IN TEACHING AND LEARNING PROBABILITY

Combinatorics is not simply a calculus tool for probability, but there is a close relationship between both topics, which is why Heitele (1975) included combinatorics in his list of ten fundamental stochastical ideas which should be present, explicitly or implicitly, in every teaching situation in the stochastic curriculum. This connection is noticeable in the main probabilistic topics of the primary and secondary mathematics curriculum, so an adequate level of combinatorial reasoning is linked to the attainment of the main curricular aims.

For grades 5-8, the Curriculum and Evaluation Standards of the National Council of Teachers of Mathematics (NCTM, 1989) recommended that the curriculum should allow students to:

- Model stochastical situations by devising and carrying out experiments or simulations to determine probabilities.

- Model stochastical situations by constructing a sample space to determine probabilities.

- Appreciate the power of using a probability model by comparing experimental results with mathematical expectations.

The concept of *random experiment* is the starting point in the study of probability at these levels. Two main aspects of a random experiment, according to Hawkins et al. (1992), are the clear formulation of the experiment and the identification of all its possible outcomes (the *sample space*). When describing simple experiments it is easy to list (enumerate) all the different outcomes of the sample space, but when we increase the number of trials, the enumeration processes can become very complex, and we may prefer to compute (count) the number of events. In both cases, we are dealing with a combinatorial problem.

The connection between combinatorics and the definition of probability is clear when using the "equally likely" approach to probability (Laplace's definition), which strongly relies on combinatorial techniques. In this approach, the probability of an event *A* is defined as the fraction $P(A)=N(A)/N$, where *N* is the total number of possible outcomes and *N(A)* is the number of outcomes leading to the occurrence of *A*. According to Piaget and Inhelder (1951), if the subject does not possess combinatorial capacity, he or she is not able to use this idea of probability, except in cases of very elementary random experiments. The Curriculum and Evaluation Standards of the NCTM (1989) recommended the following (combinatorial) procedures for students to obtain these mathematically derived probabilities: building a table or tree diagram, making a list, and using simple counting procedures.

The *frequentist* approach to probability is based on experiments and simulations. Young children may use manipulative materials to determine experimental or empirical probability. By actually conducting an experiment several times, children determine the number of ways an event occurred, and by comparing those with the total number of experiments they obtain an experimental estimate of probability. Students in grades 9-12 can also do simulations, in which urn models and computer simulation are used to describe real experiments. Urn models are based on the idea of sampling, in which the definition of the combinatorial operations might be based. Moreover, students should develop a real comprehension of the power and limitations of simulation and experimentation only by comparing experimental results to mathematically derived probabilities, which frequently require combinatorial reasoning.

For grades 5-8, some examples of compound experiments, which are linked to combinatorial operations, are also suggested in the Standards of the NCTM (1989). The inventory of all the possible events in the sample space of a compound experiment requires a combinatorial constructive process from the elementary events in the single experiments. On the other hand, arrangements and combinations may be defined by means of compound experiments (ordered sampling with/without replacement or non-ordered sampling with/without replacement).

For grades 9-12, new probabilistic topics are included, in particular, discrete probability distributions. These distributions are expressed in many cases by means of combinations or permutations, as is the case of binomial or hypergeometric distributions.

To finish this section off, we recall that many probability misconceptions are related to the lack of combinatorial reasoning that often provokes the erroneous enumeration of the sample space in the problem. This suggests the need to help students develop their combinatorial capacity.

In the next section we shall analyze the main points on which the teaching and assessment should be focused. We use the following notation: $AR_{m,n}$ for the arrangements with repetition of *m* things, taken *n* at a time (respectively, *Am,n* for the arrangements without repetition, $CR_{m,n}$ for the combinations with repetition, and $C_{m,n}$ for the ordinary combinations).

STUDENTS' COMBINATORIAL REASONING AND ITS ASSESSMENT

Besides its importance in developing the idea of probability, combinatorial capacity is a fundamental component of formal thinking. According to Inhelder and Piaget (1955), combinatorial operations represent something more important than a mere branch of mathematics. They constitute a scheme as general as proportionality and correlation, which emerge simultaneously after the age of 12 to 13 (the formal operation stage in Piagetian theory).

Combinatorial capacity is fundamental for hypothetical-deductive reasoning, which operates by combining and evaluating the possibilities in each situation. According to these authors, adolescents spontaneously discover systematic procedures of combinatorial enumeration, although for the permutations, it is necessary to wait until they are 15 years old.

More recent results, such as Fischbein's (1975), show that combinatorial problem solving capacity is not always reached, not even at that late age, without specific teaching. However, Fischbein and Gazit (1988) studied the effect of specific instruction on combinatorial capacity, and discovered that even 10-year-old pupils can learn some combinatorial ideas with the help of the tree diagram. Engel et al. (1976) have also been successful in teaching combinatorics to very young children and describe different games and activities to introduce this topic.

The role of problem solving in assessment

According to recent trends in mathematics education, mathematics is not just a symbolic language and a conceptual system, but mainly a human activity involving the solution of socially shared problems. The vision of the Curriculum and Evaluation Standards is that mathematical reasoning, problem solving, communication, and connections must be central in teaching and assessment. As stated in Garfield (1994), it is no longer appropriate to assess students' knowledge by having students compute answers and apply formulas. According to Romberg (1993), an authentic assessment should be developed by determining the extension in which the student has increased his/her ability to solve non-routine problems, reason, communicate, and apply mathematical ideas in a variety of related problems.

Consequently, both the teaching and assessment of combinatorics should be based on solving different combinatorial problems in which students need systematic enumeration procedures, recurrence, classification, tables, and tree diagrams. For example, let us consider the following item taken from Green's research concerning 11 to 16 year-old students' probabilistic reasoning (Green, 1981):

Example 2:

> Three boys are sent to the headmaster for cheating. They have to line up in a row outside the headmaster's room and wait for their punishment. No one wants to be first, of course! Suppose the boys are called Andrew, Burt, and Charles (A, B, C for short). We want you to write down all the possible orders in which they could line up. How many ways can the boys be lined up in?

Some students may use a tree diagram to write down all the solutions. In other cases, they may locate the boy to be lined up in first place (for example, Andrew), so that they reduce the problem to listing all the permutations of the two remaining boys, using recurrence (solving the problem with the help of a simpler version of the original problem). Even if some students proceed by trial and error, by writing down the possible permutations of the three boys without any systematic procedure, they may try to classify all the permutations produced according to the boy lined up in the first place, to check whether there is any forgotten permutation.

In this item, Green increased to four and five the number of boys to be lined up, to check whether students used recurrence and generalized their first solution to new parameter values. Using this item with our students, we found a typical mistake, consisting of giving solutions 8 and 10 for the permutations of four and five elements, after finding the correct number of permutations for using three-elements enumeration. Such reasoning is similar to that of the following student:

Teacher: You have written ABC/ BCA/ CAB/ ACB/ BAC/ CBA for the different ways the three boys may be lined up. Why do you think there are just 8 different ways for the four boys to be lined up?

Student 1: As I have one more boy, I can put him in first or last place. So I have six and two that is equal to eight different ways.

Teacher: I see..., but, What about the five boys?

Student 1: Because you have eight ways with four boys, when you add another new boy, you can place him in first or last place, so you must add two to the eight possible ways and you obtain ten different ways.

To solve counting problems, students could start by enumerating some cases to discover the problem structure with or without the help of a tree diagram. They might use multiplication to count grouped collections instead, by formulating and applying addition and multiplication rules. As an example, we shall analyze the solution given by a 14-year-old girl, who had not been taught combinations and arrangement formulae, to the following counting problem:

Example 3:

Four children: Alice, Bert, Carol, and Diana go to spend the night at their grandmother's home. She has two different rooms available (one on the ground floor and another upstairs) in which she could put some or all the children to sleep. In how many different ways can the grandmother put the children into the two different rooms? (She could put all the children in one room). For example, she could put Alice, Bert and Carol into the ground floor room and Diana in the upstairs room.

This girl (Jessica) enumerated all the ordered decomposition of the number 4 into two addends (a partition of the four children in two groups), that is, 4=4+0 = 3+1= 2+2=1+3=0+4. Then she counted the number of possibilities for each of these decompositions, and, finally, she applied the addition rule. If the students had been taught the combinatorial operation, they might have recognized the combinatorial operation that is the solution to the problem, here $AR_{2,4}$,i.e., the arrangements with replacement of two elements (the available rooms) taken four at a time (one room for each child).

Besides solving verbal problems, it is possible to ask students to prove combinatorial statements or to generalize the solutions found for a given problem. For example, we could ask the students to generalize the solution to Example 2, when there were *n* boys to be lined up. Both instruction and assessment should emphasize combinatorial reasoning as opposed to the application of analytic formulae for permutations and combinations. Instead of establishing the identity $C_{n,r}= C_{n,n-r}$ by algebraic manipulations**,** it is preferable for the students to reason that if you take a sample of *r* objects from *n* given objects there are still *n-r* objects left. If we encourage students to formulate their own problems, we also will improve the quality of instruction as recommended in the NCTM (1991). On the other hand, Gal and Ginsburg (1994) noticed that the creation of a problem-solving environment requires an emotionally supportive atmosphere, where students feel safe to explore, are motivated to work longer, feel comfortable with temporary mistakes and are not afraid to apply different tools to the same problem, so we should try not to be too rigid in our teaching methods.

CLASSIFICATION OF COMBINATORIAL PROBLEMS AND IMPLICATIONS FOR TEACHING AND ASSESSMENT

As Webb (1993) stated, the interpretation of a student's responses implies making inferences about what a student knows. The items in the assessment instruments form a sample of the possible tasks concerning a specific concept or procedure, so obtaining more accurate inferences requires drawing as much varied information as possible. In this section, we analyze the main task variables in combinatorial problems in order to help teachers when selecting representative samples of problems for teaching and assessment purposes.

Implicit mathematical model in simple combinatorial problems

According to Dubois (1984), simple combinatorial configurations may be classified into three models: *selections*, which emphasize the concept of sampling; *distributions*, related to the concept of mapping; and *partition* or division of a set into subsets.

The selection model

In the *selection model*, a set of m (usually distinct) objects are considered, from which a sample of n elements must be taken, as asked in the following problem (Fischbein & Gazit, 1988):

Example 4:

> There are four numbered marbles in a box (with the digits 2, 4, 7, 9). We choose a marble and note its number. Then we put the marble back into the box. We repeat the process until we form a three-digit number. How many different three-digit numbers is it possible to obtain? For example, the number 222 is a possible combination.

The keyword "choose," included in the statement of the problem, suggests to the student the idea of sampling marbles from a box. Other key verbs that usually refer to the idea of sampling are "select," "take," "draw," "gather," "pick," etc.

For the student of probability it is easy to model counting methods by performing n drawings of m numbered balls from an urn. In selecting a sample, sometimes students are allowed to repeat one or more elements in the sample, as in example 4, and other times they are not. According to this feature and whether the order in which the sample is drawn is relevant (example 4) or not, we obtain four basic sampling procedures: a) with replacement and with order ($AR_{m,n}$), b) with replacement and without order ($CR_{m,n}$), c) without replacement and with order ($A_{m,n}$) and d) without replacement and without order *($C_{m,n}$)* (permutations are a particular case of arrangements when m=n).

The distribution model

A second type of problem refers to the *distribution* of a set of n objects into m cells, such as in the following problem, in which each of the three identical cards must be introduced (placed) into one of four different envelopes (Batanero et al., to be published):

Example 5:

> Supposing we have three identical letters, and we want to place them into four different colored envelopes: yellow, blue, red, and green. It is only possible to introduce one letter into each different envelope. How many ways can the three identical letters be placed into the four different envelopes? For example, we could introduce a letter into the yellow envelope, another into the blue envelope, and the last letter into the green envelope.

Other key verbs that could be interpreted in the distribution model are "place," "introduce," "assign," "store," etc. The solution to this problem is $C_{4,3}$, but there are many different possibilities in this model, depending on the following features:

- Whether the objects to be distributed are identical (as in this problem) or not.
- Whether the containers are identical or not, as in the example.
- Whether we must order the objects placed into the containers (this makes no sense in example 5 since the objects are identical).
- The conditions that you add to the distribution, such as the maximum number of objects in each cell, or the possibility of having empty cells and so on. (In the problem proposed you may only introduce one letter into each envelope and there is an envelope left empty, but these conditions could be changed.)

Assigning the n objects to the m cells is equivalent, from a mathematical point of view, to establishing an application from the set of the n objects to the set of the m cells. For injective applications we obtain the arrangements; in the case of a bijection we obtain the permutations. Nevertheless, there is no direct definition for the combinations using the idea of application. Moreover, if we consider a non-injective application, we could obtain a problem for which the solution is not a basic combinatorial operation, so there is not a different combinatorial operation for each different possible distribution. For example, if we consider the non-ordered distribution of n different objects into m identical cells, we obtain the second kind Stirling numbers $S_{n,\ m}$. Consequently, it is not possible to translate each distribution problem into a sampling problem. The reader may find a comprehensive study of Stirling numbers in Grimaldi (1989) and for the different possibilities in the distribution model in Dubois (1984).

The partition model

Finally, we might be interested in splitting a set of n objects into m subsets, i.e., performing a *partition* of the set, as in the following problem (Batanero et al., in press):

Example 6:

> Mary and Cindy have four stamps numbered from 1 to 4. They decide to share the stamps, two for each of them. In how many ways can they share the stamps? For example, Mary could keep the stamps numbered 1 and 2 and Cindy the stamps numbered 3 and 4.

We may visualize the distribution of *n* objects into *m* cells as the partition of a set of *m* elements into *n* subsets (the cells).Therefore, there is a bijective correspondence between the models of partition and distribution, though for the student this may not be evident. Other key verbs associated with partition are “divide,” “distribute,” “split,” “decompose,” “separate,” etc.

In our research, (Batanero et al., to be published) we showed that the three types of problems we have described (selections, distributions, and partition) are not equivalent in difficulty for the students, even after being taught combinatorics. Other task variables that have affected students’ responses in Fischbein and Gazit's research (1988) are the combinatorial operation involved in the problem (combination, permutation, or arrangement), the sizes of parameters *m* and *n*, and the type of element to be combined (letter, numbers, people, objects).

Consequently, all these problem features should be considered as fundamental task variables in teaching and assessing combinatorics. As regards the sampling model, Hawkins et al. (1992) suggested that the attempt to describe a particular combinatorial problem by one of the sampling models will force the student to look carefully at the mechanism underlying random experiments. This proposal ought to be extended to the distribution and partition models. Suggesting that students conceptualize probabilistic and combinatorial problems using these three prototype models (selection, distribution, and partition) may not guarantee that the counting will be correct, but it will prevent some probabilistic misconceptions amongst the students. It may also help students to develop probabilistic reasoning, problem solving, heuristic strategies, communication and connections with other mathematical ideas, suggested as central points in the mathematics curriculum.

ASSESSING STUDENTS’ DIFFICULTIES IN SOLVING COMBINATORIAL PROBLEMS

New assessment approaches are intended to better capture how students think, reason, and apply their learning. This requires focusing the problem of assessing mathematical knowledge from a new perspective, as “the comprehensive accounting of an individual's or group's functioning within mathematics or in the application of mathematics” (Webb, 1992, p. 662). The goal is assessing the implied processes and not only measuring the degree to which students have acquired a given content. A wider range of measures, most of them qualitative ones, would be needed (Romberg et al., 1991). Assessment is not the aim of educational experiments but rather a continuous and dynamic process that can be used by teachers to help students attain curricular goals. Therefore, a key point in assessing combinatorial reasoning is identifying the students' difficulties in solving combinatorial problems, some of which shall be described in this section.

Non-systematic enumeration

This difficulty consists of trying to solve the problem by enumeration using trial and error, without any recursive procedure leading to the formation of all the possibilities. Consider, for example, the interview with student 2 (15-years-old, who had not studied combinatorics yet) to explain his solution to the following problem (Batanero et al., 1994):

Example 7:

The garage in Angel's building has five spaces numbered from 1 to 5. Because the building is very new, there are only three different residents: Angel, Beatrice, and Carmen (A, B, and C) who park their cars in the garage. For example, Angel could park his car in place number 1, Beatrice in place number 2, and Carmen in place number 4. In how many different ways could Angel, Beatrice, and Carmen park their cars in the garage?

Teacher: How would you solve this problem?

Student 2: We have three cars: A, B, C, don’t we? So I can park Angel’ s car in the first space, Beatrice’ s car in the second, and Carmen’s in the third, so I write down A=1, B=2, C=3. Then, another position could be that I put Angel’s car in the second space: A=2, B=3, C=1, or, perhaps,

A=3, B=1, C=2;
A=1, B=3, C=2;
A=2, B=1, C=3;
A=4, B=3, C=1;
A=3, B=1, C=4;
A=1, C=4, B=3;
A=3, C=4, B=1.

Teacher: Do you think there is any other possibility?

Student 2: I don’t know... I suppose I could continue in different positions, because I haven’t used 1, 2 and 4 yet, and I could change the order. What I mean is that it is not the same thing when Angel put his car in place number 2 as when it is Carmen who parks her car in space number 2. Do you see what I mean?

In spite of understanding the type of combinatorial configuration he was asked to produce, this student was unable to find all the different possibilities, because he did not follow a systematic procedure. To assess if he could solve the problem with a smaller value of the parameters, we asked him to solve the same problem with only two people (Angel and Beatrice) and three spaces in the garage. Below we reproduce what the student wrote as the solution to this new problem:

A=1, B=2;
A=2, B=1;
A=1, B=3;
A=3, B=1;
A=2, B=3;
A=3, B=2.

With this smaller number of elements the student followed a system. Nevertheless, he was unable to use recurrence to link the original problem of parking three cars with the solution obtained for this simpler version.

Incorrect use of the tree diagram

Tree graphs are one of the most useful resources for visualizing both combinatoric and probabilistic situations. In Fischbein's terminology they belong to "diagramatic models" and present important intuitive characteristics. They offer a global representation of the situation structure and this contributes to the immediacy of understanding and to finding the problem solution. In spite of this importance, Pesci (1994) proved that students found it difficult to build suitable tree diagrams to represent problem situations and, so, the same graph is the cause of many errors.

Error of order

This mistake consists of confusing the criteria for combinations and arrangements, i.e., distinguishing the order of the elements when it is irrelevant or on the contrary, not considering the order when it is essential. Here is one example taken from a student's written solution to the following example.

Example 8:

In how many ways can a teacher select three pupils to rub out the blackboard, if five students (Elisabeth, Ferdinand, George, Lucy, and Mary) have offered to do it?

E= Elisabeth, F= Ferdinand, G= George, L= Lucy, and M=Mary
E F G, E F L, E F M, E G F, E G L, E G M, E LM, E L F, E L G, E M F, E M G, E M L;
12 x5= 60; therefore, you have 60 different ways.

Error of repetition

The student does not consider the possibility of repeating the elements when it is possible, or he/she repeats the elements when there is no possibility of doing so. This is an example in a student's written answer to the following item, adapted from Fischbein and Gazit (1988), in which we also note the lack of systematic enumeration:

Example 9 :

In an urn there are three marbles numbered with the digits 2, 4, and 7. We extract a marble from the urn and note its color. Without replacing the first marble, we extract another one and note its number. Finally, we extract the last marble from the urn. How many three-digit numbers can we obtain with this method? For example, we could obtain the number 724.

724, 742, 722, 772, 744, 472, 427, 477, 444, 422, 274, 247, 277, 222, 244;
you can have 15 different numbers.

Confusing the type of object

This type of error occurs when students consider that identical objects are distinguishable or that different objects are indistinguishable. Below we reproduce the interview that we carried out with a student about his solution to the following problem (Fischbein et al., 1970):

Example 10:

> Each one of five cards has a letter: A, B, C, C and C. In how many different ways can I form a row by placing the five cards on the table? For example, I could place the cards in the following way: ACBCC.

Student 3: It is a permutation of five elements without repetition.
Teacher: Why do you think it is a permutation?
Student 3: I do not remember very well... It is the same thing as when you need to place five books on a shelf, because you cannot repeat the book. The different ways in which you may line up the letters. That would be permutation.

The teacher asked the student to write down all the different possibilities for placing the three letters ACC on the table, thereby reducing the size of the parameters in order to simplify the problem and to better understand the student's reasoning. The student started writing:

Student 3: AC1C2.
Teacher: Why do you write C1C2? I wanted you to use A, C and C!
Student 3: But I need to differentiate between the two Cs, because you have two different cards, although each has the same letter C. So, these are all the possibilities:
AC1C2, AC2C1, C1AC2, C2AC1, C1C2A, C2C1A; there are six in total.

Confusing the type of cell (the type of subsets) in partition or distribution models

This mistake consists of believing that we could distinguish identical (subsets) cells, or that it is not possible to differentiate the distinguishable cells (subsets). For example, in the following item some students only consider the three different ways in which the set of the four students can be divided into two groups. So they do not differentiate which group was going to complete the mathematics project and which was going to undertake the language project.

Example 11:

> Four friends: Ann, Beatrice, Cathy, and David, must complete two different projects: one in mathematics, and the other one in language. They decided to split up into two groups of two pupils, so

that each group could perform a project. In how many different ways can the group of four pupils be separated to perform these projects? For example, Ann and Cathy could complete the mathematics project, and Beatrice and David the language project.

Misunderstanding the type of partition required

This can occur in the following two ways: The union of all the subsets in a partition does not contain all the elements of the total set, or some possible partitions are forgotten. We can observe these two errors in the answer provided by a student to the following problem, in which he only considered two types of partitions: giving all the cars to one child or giving only one car to each child.

Example 12:

A girl has four different colored cars (black, orange, white, and gray) and she decides to share the cars with her brother John, and her sisters Peggy and Linda. In how many different ways can she share the cars? For example, she could give all the cars to Linda.

Student 4: black, orange, white, gray for Peggy;
black, orange, white, gray for John;
black, orange, white, gray for Linda;
black for Peggy; black for John; black for Linda;
orange for Peggy; orange for John; orange for Linda;
white for Peggy; white for John; white for Linda;
gray for Peggy; gray for John; gray for Linda.

IMPLICATIONS

In this chapter we have shown that combinatorial reasoning is not restricted to solving verbal combination and arrangement problems, but that it includes a wide range of concepts and problem-solving abilities. Most of these components are fundamental tools in developing probabilistic reasoning and in attaining the curricular probabilistic goals for primary and secondary education. With the help of manipulative materials and tree diagrams, meaningful activities linked to probability may be proposed, even to very young children. These activities may also serve to develop and assess problem solving and communication skills and connections to other mathematical topics.

Some of the task variables we have described in this chapter, especially the implicit mathematical model, have shown their strong effect on both the difficulties of the combinatorial problems and the types of errors in different research work. Consequently, we need to consider these task variables when we assess students' combinatorial reasoning if we want to get a more comprehensive idea of students' capabilities and conceptions.

These variables also need to be recognized when organizing our teaching, which should also emphasize the modeling process, the recursive reasoning and the systematic procedures of enumeration, instead of merely concentrating on algorithmic aspects and on the definitions of the combinatorial operations.

We have also presented some examples of tasks used to assess combinatorial reasoning in different experimental research, which we proposed to students in written questionnaires or in interviews with and without the help of manipulative material. Other teachers may want to use our examples to build their own items for teaching or assessment purposes, changing the values for task variables as needed. Or, our various examples could be included in different assessment methods, such as questions, exams, homework, portfolios, interviews, classroom discussion, and individual or collective projects (Garfield, 1994). As stated by Webb (1993), any form of assessment includes not just the task, but the students' responses, the interpretation of these responses, the meaning given to them, and the report on the assessment findings.

The careful reporting and analysis of our students' responses is an essential part of our success as teachers. Of particular benefit to our understanding of student difficulties with combinatorial reasoning is the classification of such responses into clearly defined categories. A thorough appreciation of the information in this chapter will assist us as teachers as we assign suitable meanings to our students' progress in combinatorics and probability.